The Quantum Vacuum Near Time-Dependent Dielectrics

Submitted by Samuel Lloyd Bugler-Lamb to the University of Exeter as a thesis for the degree of Doctor of Philosophy in Physics

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University. Parts of the work discussed in this thesis are in included in papers [1–3]

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Abstract

The vacuum, as described by Quantum Field Theory, is not as empty as classical physics once led us to believe. In fact, it is characterised by an infinite energy stored in the ground state of its constituent fields. This infinite energy has real, tangible effects on the macroscopic clusters of matter that make up our universe.

Moreover, the configuration of these clusters of matter within the vacuum in turn influences the form of the vacuum itself and so forth. In this work, we shall consider the changes to the quantum vacuum brought about by the presence of time-dependent dielectrics. Such changes are thought to be responsible for phenomena such as the simple and dynamical Casimir effects and Quantum Friction.

After introducing the physical and mathematical descriptions of the electromagnetic quantum vacuum, we will begin by discussing some of the basic quasi-static effects that stem directly from the existence of an electromagnetic ground state energy, known as the *zero-point energy*. These effects include the famous Hawking radiation and Unruh effect amongst others.

We will then use a scenario similar to that which exhibits Cherenkov radiation in order to de-mystify the 'negative frequency' modes of light that often occur due to a Doppler shift in the presence of media moving at a constant velocity by showing that they are an artefact of the approximation of the degrees of freedom of matter to a macroscopic permittivity function. Here, absorption and dissipation of electromagnetic energy will be ignored for simplicity.

The dynamics of an oscillator placed within this moving medium will then be considered and we will show that when the motion exceeds the speed of light in the dielectric, the oscillator will begin to absorb energy from the medium. It will be shown that this is due to the reversal of the 'radiation damping' present for lower

velocity of stationary cases.

We will then consider how the infinite vacuum energy changes in the vicinity, but outside, of this medium moving with a constant velocity and show that the presence of matter removes certain symmetries present in empty space leading to transfers of energy between moving bodies mediated by the electromagnetic field.

Following on from this, we will then extend our considerations by including the dissipation and dispersion of electromagnetic energy within magneto-dielectrics by using a canonically quantised model referred to as 'Macroscopic QED'. We will analyse the change to the vacuum state of the electromagnetic field brought about by the presence of media with an arbitrary time dependence. It will be shown that this leads to the creation of particles tantamount to exciting the degrees of freedom of both the medium and the electromagnetic field. We will also consider the effect these time-dependencies have on the two point functions of the field amplitudes using the example of the electric field.

Finally, we will begin the application of the macroscopic QED model to the path integral methods of quantum field theory with the purpose of making use of the full range of perturbative techniques that this entails, leaving the remainder of this adaptation for future work.

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Chapter I

Background

I.1 Introduction

Imagine a perfect vacuum: completely empty space, devoid of all matter except for two perfectly smooth, perfectly reflecting mirrors of infinite area, aligned perfectly parallel to one another with *zero* relative momentum, and at a temperature of absolute zero. A perfectly stationary scenario. However, as time evolves, the plates slowly begin to move towards each other. Stationary no more.

The question that immediately comes to mind is, what is the origin of the attractive force responsible for this motion? The answer is complex, but the short answer is *a change in the quantum vacuum*.

To understand this *Casimir* force [4, 5], as it is now known, consider taking this seemingly classical scenario and "*zooming*" in closer and closer towards the scale of the constituent particles of the plates; one would begin to ascertain the origin of this attractive force. The individual particles of these parallel plates would all be oscillating, perfectly randomly. Each electron, orbiting each nucleus, being pulled and pushed in all directions. This random *dance of the charges* conspires to create a net attractive force between the effective centres of mass of the two plates. The question then becomes, why are these charges vibrating at all? What is the origin of this motion? These particles do not reside in the realm of classical physics and one is no longer necessarily required to turn to external forces to explain movement. In fact, '*stationary*' does not really exist at the scale of quantum mechanics. In the simple picture described, the fundamental fluctuations of quantum mechanics are responsible for this dance of the charges that leads to the macroscopic attractive force that acts at a distance far greater than the scale of the very fluctuations to which is owes its existence. The form that these vacuum fluctuations take will be the main focus of this work.

Although they act on a scale much larger than the scale of the quantum world, these forces are obscured at the scale of most human activities. Generally weak, they are usually most apparent where the scale of the interacting bodies are of the order of nanometres ($\approx 10^{-9}\,m$).

Although the precision of Casimir's original prediction [4, 6, 7] has been disputed, its success lies in the simplicity of the model and its ability to capture the essence of the phenomena with very few superfluous considerations. In his landmark paper [4], his reasoning rests on the simple fact that for a given distance d_1 between the mirrors, one can compute the electromagnetic energy density in between the mirrors and on either side of the two mirror set-up which originates from the ground state of the electromagnetic field. For reasons that will be explored in due course, this energy turns out to be infinite. For a different spacing d_2, the energy between the reflectors will again be infinite but this time, the electromagnetic field will be in a different configuration and this infinity will be different, as such the difference between the two energies will be non-zero, i.e $\Delta E = E(d_1) - E(d_2) \neq 0$. The result is that the system, forever striving to attain the configuration of lowest energy, will conspire to reduce the distance between the plates. Crucially, it is important to note that the relative position of the plates dictates the *form* of the electromagnetic field. It is the 'vacuum fluctuations' of this field that lead to a real measurable effect on the motion of large bodies.

Now, imagine one of the plates begins to move relative to the other at a constant speed. This could really be in any direction, however, suppose for now that its relative motion is in a direction parallel to both surfaces. How does this motion affect this intricate correlation between the constituent dipoles and eventually the Casimir force? It turns out in this particular scenario the force remains unchanged. However, if, rather than perfect mirrors, one were to use two *dispersive* dielectric plates, the most interesting consequence of the motion is that this correlation, mediated by the quantum fluctuations, conspires to pull against the relative motion itself [8–11]. This means that without a continuous external force to maintain this motion, the relative speed will gradually reduce and eventually disappear altogether. This is known as *quantum friction*; a frictional force that is a direct result of the mechanics at the quantum scale that does not involve the classical means of friction. Unlike classical friction which arises from collisions between large clusters of particles sullying the surface of relatively moving objects, leading to a generation of heat and a reduction in the relative motion, the term 'quantum friction' generally refers to effects that exist even between entities with perfectly smooth non-colliding surfaces.

Analysis of quantum friction is somewhat more intricate than the simple Casimir force and this has led to much controversy about not just the scale of quantum friction but whether or not it even exists [8, 12]. Often in such calculations, "negative frequency" modes of light arise (see for example [8]) and seem to bear some importance, but their interpretation has until now been difficult and they have remained somewhat mysterious. This phenomenon turns out to be independent of the inclusion of dissipation and dispersion and will be covered in Chapter II. In simple terms, we shall show that this is an artefact of approximating media by a permittivity function rather than taking into account its microscopic structure. It is important to reiterate, however, that the material's ability to absorb and disperse electromagnetic energy seems to be a crucial factor in frictional calculations [8–11] and as such, in Chapter III, we will consider how their inclusion modifies the vacuum field.

For the Casimir effect and Quantum Friction, it is important to stress again that each configuration of the media results in different configurations of the electromagnetic field. It is the fluctuations of this field that create these forces. Both effects are part of a larger family of physical manifestations whose origin is understood to be related to these changes in the vacuum energy.

In Chapter II, we will consider the physics of an electrically neutral oscillator travelling through a medium. This closely resembles another notable and closely related example in this family commonly referred to as Cherenkov radiation [13–20] although it was theorised many years before its experimental discovery by Heaviside [21]. Cherenkov radiation refers to the radiation emitted when a charged particle travels through a material faster than the speed of light in said material. The main difference with the case we will explore is that we will rely on the fluctuations of the vacuum field to polarise the neutral particle in order for the radiation to occur.

Furthermore, these same vacuum fluctuations are also thought to be responsible for the gradual evaporation of black holes [22–24]. Hawking radiation, as it is called, also exhibits the negative frequency modes of light that we will discuss within Chapter II. The phenomenon is sometimes explained by the picture of the vacuum field being composed of *virtual* particle pairs that appear and disappear spontaneously after some small finite time. In this picture, if a pair is created on the surface of a black hole, the one on the inside of the event horizon succumbs and the one on the outside escapes leaving an overall net radiation. This phenomena is similar in nature to that most commonly referred to as the Unruh effect [25–27] in which similar modes are observed in an accelerating reference frame within the vacuum. Here, the acceleration alone is sufficient to measure the creation of pairs of particles leading to the appearance of a thermal background. This is related somewhat to the main findings of Chapter III in which we periodically accelerate a dielectric within the vacuum.

Recently, much work has been done to provide a laboratory analogue to the now famous Hawking radiation [25, 28–33]. In the optical analogues [32, 33], a pulse like perturbation to the dielectric properties is used to simulate an event horizon. It is thought that the vacuum energy again plays a crucial role in the generation of the relevant modes of light. In Chapter III, we will consider a very similar physical picture to these optical analogues: that of a dielectric pulse travelling through an otherwise homogeneous and isotropic dielectric.

The localised distortion of the permittivity profile of dielectrics used to generate dielectric 'pulses' like this is a trick often used within a group of experiments referred to as dynamic Casimir systems [7, 34–42]. In essence, this refers to situations, very similar to Casimir's original consideration but where the permittivity is forced to change usually by the application of an external laser acting to locally polarise the material; such a situation will be considered in Chapter III. This forced change is often used to simulate the vibrational motion of macroscopic dielectrics when the frequency of oscillation is too great to be experimetnally practical [34]. However, as we will briefly explain in Chapter III, this analogy has been somewhat controversial [43, 44].

Furthermore, the fluctuating quantum vacuum field also gives rise to the Lamb Shift [45] where it modifies the energy levels of the hydrogen atom's electron, an effect not predicted before its experimental discovery by Lamb and Retherford.

In this thesis, we will focus on macroscopic time dependent dielectrics and ask how the motion of these dielectrics changes the electromagnetic vacuum field. To this end, we will begin by giving a brief overview of the classical electromagnetism relevant to this work, followed by a mathematical introduction to the quantum vacuum and its immediate effects. Then, we will consider how the electromagnetic vacuum field is modified by the presence of media moving with a constant speed without including any dispersion or dissipation of the electromagnetic energy within the medium. The following chapter will be dedicated to exploring how the vacuum field is changed by including this dissipation and dispersion for stationary media and media moving with a constant velocity, finishing up by showing the change in the vacuum by media subject to some arbitrary, time dependent velocity with a time dependent electric response function. In the final chapter, we shall explore the path integral methods that one may use to understand the form of the vacuum field for arbitrary time-dependencies of the medium.

I.2 Classical Electromagnetism

In its simplest terms, the value of the electric field $\mathbf{E}$ at some position in space-time $(\mathbf{x}, t)$ is a vector that represents a measure of the force $\mathbf{F}$ that would act upon a positive unit charge if it were to be placed at the position $\mathbf{x}$ at a time t. The force acting on some charge q would be given by $\mathbf{F} = q\mathbf{E}$. This simple definition of the electric force is valid when the charge is at rest and the system is only characterised by a static electric field.

A more general force, known as the Lorentz force, considers the case where the charge may be in motion and includes the possibility that the system is also host to a magnetic field $\mathbf{B}$. In this case, if the charge is moving with velocity $\mathbf{v}$, the force is given by $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The origin of the form of this force may at first seem obscure but clarity is found by understanding that the electric and magnetic fields are in fact two facets of the same underlying field, known as the electromagnetic potential. This potential is written in covariant form[1] as A_ν composed of a scalar potential $\phi = A_0$ and a vector potential $\mathbf{A} = (A_1, A_2, A_3)$. From this potential are built the electric and magnetic fields via the vectorial expressions

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \dot{\mathbf{A}} \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{aligned} \tag{I.1}$$

It is useful to define the Lorentz invariant electromagnetic field tensor $F_{\mu\nu}$ which is composed of the various components of the two fields via the defining expression $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Many of the calculations in this thesis will consider moving media; as such, it is useful to remember the basic results of special relativity. Special relativity describes how the fields and coordinates, amongst other quantities, transform between inertial reference frames. Consider two inertial frames, moving with a relative velocity $\mathbf{v}$, one frame shall be denoted with primes ($\mathbf{x}'$, t', $\mathbf{A}'$, etc.) that shall be the "moving frame" and one without ($\mathbf{x}$, t, $\mathbf{A}$, etc.), the "lab frame". The

[1] For an introduction to covariant notation, see for example [46].

coordinate transformation between frames are neatly related via

$$x^{\nu\prime} = \Lambda^{\nu}{}_{\mu} x^{\mu}. \tag{I.2}$$

For simplicity we write the vectors in terms of a component parallel to the velocity, denoted by the subscript $||$ and a component perpendicular to the velocity, denoted by the subscript $\perp$, i.e $x^{\mu} = \big(x_{\perp}, x_{||}, ct\big)$, the transformation matrix $\Lambda^{\nu}{}_{\mu}$ is given by

$$\Lambda^{\nu}{}_{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & -\gamma\beta \\ 0 & -\gamma\beta & \gamma \end{pmatrix} \tag{I.3}$$

and the Lorentz factor λ is given by

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{I.4}$$

where $\beta = v/c$. The derivatives, given as $\partial/\partial x^{\nu}$, transform via

$$\frac{\partial}{\partial x^{\nu\prime}} = \Lambda_{\nu}{}^{\mu} \frac{\partial}{\partial x^{\nu}} \tag{I.5}$$

where the transformation matrix for covariant quantities is instead defined as

$$\Lambda_{\nu}{}^{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & \gamma\beta \\ 0 & \gamma\beta & \gamma \end{pmatrix}. \tag{I.6}$$

The fields of course transform in a similar manner via $F'_{\mu\nu} = \Lambda_{\mu}{}^{\alpha}\Lambda_{\nu}{}^{\beta}F_{\alpha\beta}$. Depending on the frame of reference, the various components of this matrix will turn out to be either just an electric field, or a magnetic field or a combination of both. For the electric field, the mixing that led to the Lorentz force introduced above is found to be summarised by the vector expressions

$$
\begin{aligned}
\mathbf{E}'_{||} &= \mathbf{E}_{||} \\
\mathbf{E}'_{\perp} &= \gamma\left(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}\right)
\end{aligned}
\tag{I.7}
$$

and for the magnetic field, the analogous expressions are found to be

$$
\begin{aligned}
\mathbf{B}'_{||} &= \mathbf{B}_{||} \\
\mathbf{B}'_{\perp} &= \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2}\times\mathbf{E}\right).
\end{aligned}
\tag{I.8}
$$

In future chapters, we shall see that this mixing gives rise to extra terms in our Lagrangian in the case of moving media within the quantum vacuum.

Furthermore, a more perplexing, and arguably more intricate vision of the universe is hidden within expressions (I.1). Remembering that the curl of a gradient is zero, i.e $\nabla\times(\nabla f) = 0$, for any function of position f, indicates that the actual values of the four potential in (I.1) are far from unique since we can seemingly arbitrarily add terms to our four potential, providing they have the right form, namely we can make the change $\mathbf{A}\to\mathbf{A}+\nabla f$ and $\phi\to\phi-\partial f/\partial t$ for any functions f dependent on both position and time leaving the electric and magnetic fields exactly the same. The result of this is that any physical effect that depends on either the electric or magnetic field will also remain the same. This important concept is called *gauge invariance* and underpins most of modern physics. In simple terms, this is a statement of the fact that the electric and magnetic fields are more fundamental.

Now consider a large cluster of coagulated particles suspended in a vacuum interacting with some ‘external’ electric and magnetic fields. By external, one means that the fields are generated by some other collection of charges sufficiently far away that we can consider them to be part of a completely separate system, and where one can neglect a back reaction from the charges in our system on these ‘external’ fields. The charges and currents in the system can be bound (meaning they have a fixed position around which they may move to varying degrees) or free (meaning that they may travel throughout the medium with varying ease). In the presence of external fields, the bound charges and currents are displaced within the media by these fields, leading to new effective fields within matter that are referred to as the *displaced* fields $\mathbf{D}$ and $\mathbf{H}$ for the electric and magnetic field respectively (see for example [47]).

The culmination of centuries of empirical observations relate the electric and magnetic fields to these displaced fields in the beautifully compact form known as Maxwell's equations. In media, these can be written as

$$\nabla \cdot \mathbf{D} = \rho \tag{I.9}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{I.10}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{I.11}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{I.12}$$

where ρ is the free charge and $\mathbf{J}$ is the free current, neither being bound to the media.

When considering dynamical fields transmitting energy from one collection of charges to another in the form of electromagnetic waves (i.e. light), depending on the scale of the wavelength of this light, its interaction with a cluster of particles can be more or less intricate. At the scale of the individual particles, this entity appears as such: a large collection of neighbours each influencing the behaviours of its kin in an intricate dance. However, for wavelengths of light of a much larger length scale, although interacting with the same particles, the EM field effectively sees a much simplified picture. Their behaviour, although in reality influenced by the same intricate dance, can be effectively described, to varying degrees, by some block properties most commonly described by a permittivity ϵ and a permeability function μ. The internal shenanigans of the material dictate the precise form that these functions take. They may be local or non-local in both time and space which means that the polarisation or magnetisation at any point in space time may depend on the field strengths at other positions at times present or past.

For most practical applications, these large scale functions are sufficient to describe the interaction of blocks of material with the electromagnetic field. In this thesis, we shall concern ourselves only with dispersive, absorbing media whose polarisation and magnetisation only depend linearly on the field strengths. We also allow this modification to be non local in space and time. We will however, mostly concern ourselves with the temporal non-locality. This modification is represented by the form taken by the polarisation

$$\mathbf{P}(\mathbf{x},t) = \epsilon_0 \int d^3\mathbf{x}' \int_{-\infty}^{\infty} dt'\, \chi_E(\mathbf{x},\mathbf{x}',t,t')\mathbf{E}(\mathbf{x}',t') \tag{I.13}$$

and the magnetisation

$$\mathbf{M}(\mathbf{x},t) = \mu_0^{-1} \int d^3\mathbf{x}' \int_{-\infty}^{\infty} dt\, \chi_B(\mathbf{x},\mathbf{x}',t,t')\mathbf{B}(\mathbf{x}',t') \tag{I.14}$$

where $\chi_E(\mathbf{x},\mathbf{x}',t,t')$ is the electric susceptibility and $\chi_B(\mathbf{x},\mathbf{x}',t,t')$ the magnetic susceptibility. The constitutive equations $\mathbf{D}(\mathbf{x},t) = \epsilon_0\mathbf{E}(\mathbf{x},t)+\mathbf{P}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t) = \mu_0^{-1}\mathbf{B}(\mathbf{x},t) - \mathbf{M}(\mathbf{x},t)$ then take the form

$$\begin{aligned}
\mathbf{D}(\mathbf{x},t) &= \epsilon_0 \int d^3\mathbf{x}' \int_{-\infty}^{\infty} dt'\, \epsilon(\mathbf{x},\mathbf{x}',t,t')\mathbf{E}(\mathbf{x}',t') \\
\mathbf{H}(\mathbf{x},t) &= \mu_0^{-1} \int d^3\mathbf{x}' \int_{-\infty}^{\infty} dt\, \mu^{-1}(\mathbf{x},\mathbf{x}',t,t')\mathbf{B}(\mathbf{x}',t')
\end{aligned} \tag{I.15}$$

where the relative permittivity is given by $\epsilon(\mathbf{x},\mathbf{x}',t,t') = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')+\chi_E(\mathbf{x},\mathbf{x}',t,t')$ and the relative permeability is given by $\mu^{-1}(\mathbf{x},\mathbf{x}',t,t') = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t') + \chi_B(\mathbf{x},\mathbf{x}',t,t')$.

If the non-locality is constrained to be temporal, and more precisely, only depends on the difference between the present time t and some past time t', the field strengths at a point $\mathbf{x}$ only depend on the value of the fields at that point $\mathbf{x}$ in the past or present. The susceptibilities then become $\chi_{E/B}(\mathbf{x},\mathbf{x}',t,t') \rightarrow \chi_{E/B}(t-t')\theta\,(t-t')\,\delta(\mathbf{x}-\mathbf{x}')$. The field amplitudes can then be written in the frequency domain for example via

$$\mathbf{E}(\mathbf{x},t)=\int\limits_0^\infty \frac{d\omega}{2\pi}\left[\mathbf{E}(\mathbf{x},\omega)e^{-i\omega t}+\text{c.c}\right] \tag{I.16}$$

and are related to the displacement field's Fourier components via $\mathbf{D}(\omega)=\epsilon_0\epsilon(\omega)\mathbf{E}(\omega)$ and $\mathbf{H}(\omega)=\mu_0^{-1}\mu^{-1}(\omega)\mathbf{B}(\omega)$ where the Fourier components of the permittivity $\epsilon(\omega)$ and permeability $\mu(\omega)$, obey the Kramers-Kronig relations[2]. This relates the real and imaginary part of any analytic complex function; the permittivity for example must obey

$$\text{Re}\left[\epsilon(\mathbf{x},\omega)\right]=1+\frac{2}{\pi}\text{P}\int\limits_0^\infty d\omega_1\frac{\omega_1\text{Im}\left[\epsilon(\mathbf{x},\omega_1)\right]}{\omega_1^2-\omega^2} \tag{I.17}$$

where P indicates the principal part of the integral[3]. The limit where the frequency dependence and the absorption vanish will also be considered in this thesis. This is also equivalent to making the change $\epsilon(\mathbf{x},\mathbf{x}',t,t')\to\epsilon_{\text{avg}}\ \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$ in (I.15) for example.

For the particular case of linear, dispersive, absorbing media with no free charges or currents, writing the fields in terms of their Fourier components via (I.16), along with the constitutive equations (I.15), allows us to write the polarisation as

$$\mathbf{P}(\mathbf{x},t)=\epsilon_0\int\limits_0^\infty \frac{d\omega}{2\pi}\left[\chi(\mathbf{x},\omega)\mathbf{E}(\mathbf{x},\omega)e^{-i\omega t}+\text{c.c}\right] \tag{I.18}$$

where $\chi(\omega)=\epsilon(\omega)-1$ is the susceptibility of the dielectric. By substituting the Fourier transforms of (I.11) into (I.12) after multiplying by the permeability, the expression for the electric wave equation is found to be

$$\nabla\times\left[\mu^{-1}(\mathbf{x},\omega)\nabla\times\mathbf{E}(\mathbf{x},\omega)\right]-\frac{\omega^2}{c^2}\epsilon(\mathbf{x},\omega)\mathbf{E}(\mathbf{x},\omega)=0. \tag{I.19}$$

A similar equation holds for the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$.

The simplicity of this equation is not to be underestimated. Had one not been able to assume bulk properties of the interaction via a permittivity and a perme-

[2]For an elegant and concise derivation of the Kramers-Kronig relations see for example [48].
[3]See Appendix A.2 for an introduction to the principal part P.

ability, the wave equation (I.19) would take a much less appealing form

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{x}, \omega) - \frac{\omega^2}{c^2}\mathbf{E}(\mathbf{x}, \omega) = i\omega \mathbf{j}(\mathbf{x}, \omega) \tag{I.20}$$

where, crucially, the current $\mathbf{j}$ would be some very complicated function of the electromagnetic field itself and would necessarily contain information about *every particle* in the system and its behaviour, *past and present*.

This concludes our brief overview of the classical electromagnetic concepts relevant to this work. However, we are predominantly concerned with how the quantum vacuum changes due to the motion of dielectrics and thus we must now introduce the quantum description of the vacuum that we will use throughout this thesis.

I.3 The Electromagnetic Quantum Vacuum

Quantum electrodynamics has two main formalisms: the canonical approach and the path integral approach. We will be using both approaches during this thesis but for now, we will content ourselves with a very brief recapitulation of the main concepts of the canonical approach. In section IV, we will introduce the path integral method before applying it to macroscopic quantum electrodynamics. For a more detailed description of both approaches, the reader is encouraged to consult [49–53], for example. The starting point for both approaches is the action S, which for a single particle is a functional of the parameters of the system q_i,

$$S\left[q_i, \dot{q}_i\right] = \int dt\, L\left[q_i, \dot{q}_i\right] \tag{I.21}$$

where L is the Lagrangian, from which all the mechanics of the classical system can be derived [46]. Varying the action with respect to the system parameters results in the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial_\mu q_i}\right) = \frac{\partial L}{\partial q_i} \tag{I.22}$$

where repeated indices imply a sum over the index (Einstein summation). The canonical momenta of the system are defined in terms of the Lagrangian via

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{I.23}$$

and the Hamiltonian is given by the Legendre transformation

$$H = \sum_i p_i \, \dot{q}_i - L. \tag{I.24}$$

The reader is reminded that quantum field theory rests upon the fact that one can construct a field such as the electromagnetic field for example, by considering a harmonic oscillator at every point in space. The Lagrangian for one harmonic oscillator of mass m and spring constant κ is customarily defined by

$$L_{\omega\sqrt{m}}[\mathsf{X}, \dot{\mathsf{X}}] = \frac{1}{2} m \dot{\mathsf{X}}^2 - \frac{1}{2} \kappa \mathsf{X}^2. \tag{I.25}$$

It can also be expressed in a form which we will most often refer to, after the change of variables $\mathsf{X} \to \mathsf{X}/\sqrt{m}$ as

$$L_{\omega}[\mathsf{X}, \dot{\mathsf{X}}] = \frac{1}{2} \left(\dot{\mathsf{X}}^2 - \omega^2 \mathsf{X}^2 \right) \tag{I.26}$$

where the natural frequency $\omega = \sqrt{\kappa/m}$. By applying the Euler-Lagrange equations (I.22) to (I.26), one can establish the standard canonical momentum of the oscillator,

$$p_{\mathsf{X}} = \frac{\partial L}{\partial \dot{\mathsf{X}}} = \dot{\mathsf{X}}. \tag{I.27}$$

In the usual manner, one can then perform the Legendre transform to obtain the Hamiltonian

$$H = p_{\mathsf{X}} \dot{\mathsf{X}} - L = \frac{1}{2} \left(p_{\mathsf{X}}^2 + \omega^2 \mathsf{X}^2 \right). \tag{I.28}$$

The quantum mechanical description of the system is built by promoting the position and momentum quantities to operators which act upon the states of the system. These operators must obey the commutation relations which, for the harmonic oscillator are

$$\left[\hat{\mathsf{X}}, \hat{p}_{\mathsf{X}}\right] = \hat{\mathsf{X}}\hat{p}_{\mathsf{X}} - \hat{p}_{\mathsf{X}}\hat{\mathsf{X}} = i\hbar. \tag{I.29}$$

Using these definitions, the Hamiltonian operator is given by $\hat{H} = \left(\hat{p}_{\mathsf{X}}^2 + \omega^2\,\hat{\mathsf{X}}^2\right)/2$. These operators can be defined in terms of creation operators $\hat{b}^\dagger$ and annihilation operators $\hat{b}$ defined such that

$$[\hat{b}, \hat{b}^\dagger] = 1 \tag{I.30}$$

which act upon the energy eigenstates of the system such that $\hat{b}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{b}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$. The various operators can be re-expressed in the Schroedinger picture (where the time dependence of the system is contained within the states and not the operators) as

$$\hat{\mathsf{X}} = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{b} + \hat{b}^\dagger\right) \tag{I.31}$$

$$\hat{p}_{\mathsf{X}} = -i\sqrt{\frac{\hbar\omega}{2}}\left(\hat{b} - \hat{b}^\dagger\right) \tag{I.32}$$

and the Hamiltonian is written in its diagonalised form via the expression

$$\hat{H} = \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right). \tag{I.33}$$

In quantum field theory, since the mathematics is built upon there being a quantum harmonic oscillator at each point in space, the Lagrangian is usually written in terms of a Lagrangian *density* $\mathscr{L}$ defined such that $L = \int d^3\mathbf{x}\,\mathscr{L}$. For electromagnetism in the vacuum space, the *free* Lagrangian density, which we shall denote as $\mathscr{L}_0$, is given in terms of the field tensor $F_{\mu\nu}$ or the electric and magnetic fields via

$$\mathscr{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 - c^2\mathbf{B}^2\right). \tag{I.34}$$

From this Lagrangian density, the equivalent Euler-Lagrange equations can be formulated

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\mathbf{A})}\right) = \frac{\partial\mathscr{L}}{\partial\mathbf{A}}, \tag{I.35}$$

where repeated indices imply a sum over the index and the momentum of the potential field

$$\Pi_{\mathbf{A}} = \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{A}}} \tag{I.36}$$

allowing the establishment of a Hamiltonian via

$$H = \int d^3\mathbf{x} \left(\Pi_{\mathbf{A}} \cdot \dot{\mathbf{A}} - \mathscr{L} \right). \tag{I.37}$$

This time, we promote the *field amplitudes* to operators by enforcing that the *equal* time commutation relations

$$\left[\hat{\mathbf{A}}(\mathbf{x}, t), \hat{\Pi}_{\mathbf{A}}(\mathbf{x}', t) \right] = i\hbar\, \delta^{(3)}_{\perp}(\mathbf{x} - \mathbf{x}') \tag{I.38}$$

are obeyed. In quantum electrodynamics, the free field is quantised in the Coulomb gauge, which is assured by the condition that $\nabla \cdot \mathbf{A} = 0$ must hold true. This results in the field being expanded in terms of purely transverse components, and we write the vector potential operator as

$$\hat{\mathbf{A}}\left(\mathbf{x}, t\right) = \sum_{\lambda} \int \frac{d^3\,\mathbf{k}}{(2\pi)^3} \sqrt{\frac{\hbar}{2\epsilon_0 \omega\left(\mathbf{k}\right)}} \left[\mathbf{e}^{\lambda}\left(\mathbf{k}\right) e^{i[\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t]} \hat{a}_{\lambda}\left(\mathbf{k}\right) + \text{h.c} \right] \tag{I.39}$$

where the frequency $\omega\left(\mathbf{k}\right) = c\left|\mathbf{k}\right|$ satisfies the dispersion relation and $\mathbf{e}^{\lambda}\left(\mathbf{k}\right)$ are the unit vectors for the mode with wave-vector $\mathbf{k}$ and polarisation λ. Note the relations $\mathbf{e}^{\lambda}\left(\mathbf{k}\right) \cdot \mathbf{e}^{\lambda'}\left(\mathbf{k}\right) = \delta_{\lambda\lambda'}$, $\hat{\mathbf{k}} \cdot \mathbf{e}^{\lambda}\left(\mathbf{k}\right) = 0$ and $\hat{\mathbf{k}} \times \mathbf{e}^{1}\left(\mathbf{k}\right) = \mathbf{e}^{2}\left(\mathbf{k}\right)$. Using the usual definitions for the electric and magnetic fields, in this gauge, in terms of the vector potential $\mathbf{E} = -\partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, leads to expressions for the electric field operator

$$\hat{\mathbf{E}}\left(\mathbf{x}, t\right) = \mathrm{i} \sum_{\lambda} \int \frac{d^3\,\mathbf{k}}{(2\pi)^3} \sqrt{\frac{\hbar\omega\left(\mathbf{k}\right)}{2\epsilon_0}} \left[\mathbf{e}^{\lambda}\left(\mathbf{k}\right) e^{i[\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t]} \hat{a}_{\lambda}\left(\mathbf{k}\right) - \text{h.c} \right] \tag{I.40}$$

and of the magnetic field operator

$$\hat{\mathbf{B}}\left(\mathbf{x}, t\right) = \mathrm{i} \sum_{\lambda} \int \frac{d^3\,\mathbf{k}}{(2\pi)^3} \sqrt{\frac{\hbar}{2\epsilon_0 \omega\left(\mathbf{k}\right)}} \left[\mathbf{k} \times \mathbf{e}^{\lambda}\left(\mathbf{k}\right) e^{i[\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t]} \hat{a}_{\lambda}\left(\mathbf{k}\right) - \text{h.c} \right] \tag{I.41}$$

where the bosonic creation and annihilation operators satisfy the commutation relations

$$\begin{aligned}
&\left[\hat{a}_{\lambda_1}\left(\mathbf{k}_1\right), \hat{a}^{\dagger}_{\lambda_2}\left(\mathbf{k}_2\right)\right] = (2\pi)^3 \delta\left(\mathbf{k}_1 - \mathbf{k}_2\right) \delta_{\lambda_1\lambda_2} \\
&\left[\hat{a}_{\lambda_1}\left(\mathbf{k}_1\right), \hat{a}_{\lambda_2}\left(\mathbf{k}_2\right))\right] = 0 \\
&\left[\hat{a}^{\dagger}_{\lambda_1}\left(\mathbf{k}_1\right), \hat{a}^{\dagger}_{\lambda_2}\left(\mathbf{k}_2\right)\right] = 0.
\end{aligned} \tag{I.42}$$

Note that $\hat{a}_{\lambda_1}\left(\mathbf{k}_1\right)|0\rangle = 0$ and $\hat{a}^{\dagger}_{\lambda_1}\left(\mathbf{k}_1\right)|0\rangle = |\mathbf{k}_1\rangle$. In this simple vacuum case, the Hamiltonian operator for the system is found to be

$$\hat{H} = \frac{\epsilon_0}{2}\left(\hat{\mathbf{E}} \cdot \hat{\mathbf{E}} + c^2\, \hat{\mathbf{B}} \cdot \hat{\mathbf{B}}\right) \tag{I.43}$$

from which one can express the vacuum expectation value of the Hamiltonian

$$\left\langle \hat{H} \right\rangle = \frac{\epsilon_0}{2}\left[\left\langle \hat{\mathbf{E}} \cdot \hat{\mathbf{E}} \right\rangle + c^2 \left\langle \hat{\mathbf{B}} \cdot \hat{\mathbf{B}} \right\rangle\right]. \tag{I.44}$$

Since we will often be working in the vacuum, we will use the simplified notaton $\langle\rangle \equiv \langle 0| |0\rangle$. Substituting in the expressions (I.40) and (I.41) into the first term of (I.44) and applying the commutation relations (I.42) leads to

$$\left\langle \hat{H} \right\rangle = \frac{\hbar}{4}\sum_{\lambda}\int d^3\mathbf{k}\,\left\{\omega\left(\mathbf{k}\right)\mathbf{e}^{\lambda}\left(\mathbf{k}\right) \cdot \mathbf{e}^{\lambda}\left(\mathbf{k}\right) + \frac{c^2}{\omega\left(\mathbf{k}\right)}\left[\mathbf{k} \times \mathbf{e}^{\lambda}\left(\mathbf{k}\right)\right] \cdot \left[\mathbf{k} \times \mathbf{e}^{\lambda}\left(\mathbf{k}\right)\right]\right\} \tag{I.45}$$

which, using $\left[\mathbf{k} \times \mathbf{e}^{\lambda}\left(\mathbf{k}\right)\right] \cdot \left[\mathbf{k} \times \mathbf{e}^{\lambda}\left(\mathbf{k}\right)\right] = \mathbf{k} \cdot \mathbf{k}$ and $\sum_{\lambda}\mathbf{e}^{\lambda}\left(\mathbf{k}\right) \cdot \mathbf{e}^{\lambda}\left(\mathbf{k}\right) = 2$, can be shown give us the vacuum energy expectation value

$$\left\langle \hat{H} \right\rangle = \int d^3\mathbf{k}\,\hbar\omega\left(\mathbf{k}\right) \tag{I.46}$$

which is infinite as the integral extends over all modes. The reader will notice that it can be easily traced back to the finite ground state energy contained in the quantum mechanics of the harmonic oscillator. Since there are infinitely many harmonic oscillators at every point in space, there ends up being an infinite energy at each point.

In future chapters, we will not just concern ourselves with the vacuum energy but will also consider the energy flows and the stress tensor. These can be derived directly from the Lagrangian density (I.34). The form of the electromagnetic stress-energy tensor according to Noether's theorem is found by varying the action [46], and each term $T^{\mu\nu}$ is defined in terms of the electromagnetic field tensor $F^{\mu\nu}$ as

$$T^{\mu\nu} = \mu_0^{-1}\left[F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\,\eta^{\mu\nu}\right] \tag{I.47}$$

where $\eta^{\mu\nu}$ is the Minkowski metric tensor $\eta = \mathrm{diag}(-1,1,1,1)$. Here we express it as

$$\mathbf{T} = \begin{bmatrix} \rho & \frac{1}{c}\mathbf{S} \\ \frac{1}{c}\mathbf{S} & -\boldsymbol{\sigma} \end{bmatrix} \tag{I.48}$$

where the energy density ρ is given by

$$\rho = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right), \tag{I.49}$$

the Poynting vector $\mathbf{S}$ is defined by

$$\mathbf{S} = \mu_0^{-1}\,\mathbf{E}\times\mathbf{B}, \tag{I.50}$$

and Maxwell's stress tensor is defined by

$$\boldsymbol{\sigma} = \epsilon_0\left[\mathbf{E}\otimes\mathbf{E} + c^2\mathbf{B}\otimes\mathbf{B}\right] - \rho\,\mathbb{1}_3, \tag{I.51}$$

where $\otimes$ is the tensor product. During this thesis, we will often concern ourselves either with the vacuum or the thermal expectations of these quantities. The expectation of the energy stress tensor is given in terms of the field operators by $\left\langle \hat{\mathbf{T}} \right\rangle$ in similar fashion to the Hamiltonian operator (I.46).

In addition to this mathematical description of the vacuum field afforded by quantum electrodynamics, we will now briefly discuss a couple of other approaches to the quantum vacuum field to which we will occasionally refer throughout this thesis.

I.4 Alternative Descriptions of the Vacuum

The vacuum fields considered in this thesis arise naturally in quantum electrodynamics; here however, we present a couple of alternative mathematical approaches which recover the same results but with a somewhat different formalism. Although, we will not directly use these methods in establishing our results, they were of great use in trying to interpret the physics underlying these phenomena and we briefly present them here for general interest and understanding.

I.4.1 Stochastic Electrodynamics

A popular alternative interpretation of the vacuum field is a line of thought that rests upon the realisation, first explored by Einstein, Plank and Hopf [54], that at zero temperature, the universe seems to be essentially governed by classical electrodynamics subject to a completely random field. This concept was later considered in more detail by Braffort [55, 56] and Marshall [57, 58], later further developed by Boyer [59], de la Pena and Cetto (see for example [60] and references therein).

Essentially, to account for the fluctuations of the vacuum field, a random term $\theta(\mathbf{k}, \lambda)$ is introduced to the phase of every mode of the EM field, where $\mathbf{k}$ is the wave vector of the mode and λ is the polarisation when the electric and magnetic fields can be represented by

$$\mathbf{E}(\mathbf{x}, t) = \text{Re}\left[\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathbf{e}_\lambda(\mathbf{k})\, g(\omega_\mathbf{k}) e^{-i\phi}\right] \tag{I.52}$$

and

$$\mathbf{B}(\mathbf{x}, t) = \text{Re}\left[\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hat{\mathbf{k}} \times \mathbf{e}_\lambda(\mathbf{k})\, g(\omega_\mathbf{k}) e^{-i\phi}\right] \tag{I.53}$$

respectively where the phase $\phi = \mathbf{k}\cdot\mathbf{x} - \omega_\mathbf{k} t + \theta(\mathbf{k}, \lambda)$ contains the random part and the polarisation vectors obey the same orthogonality relations as in section I.5. The function $g(\omega_\mathbf{k})$ is then found by enforcing that the vacuum energy be invariant under a Lorentz transformation. A possible function that satisfies this condition ends up retrieving the usual vacuum energy of quantum field theory, i.e. $g(\omega_{\mathbf{k}}) = \sqrt{\hbar\omega/2\pi^2}$. Here, it can be seen that the Plank constant appears as a factor that sets the scale. The ultimate goal of this theory was to make sense of this zero point energy. However, it seems unclear from the literature whether a consensus has been reached on the supposed origin of this random field.

Practically speaking, the effect of the random phase arises when averaging over modes where we end up introducing correlations between the phases of the fields leading to terms like $\langle \cos(\phi_1)\cos(\phi_2)\rangle = 1/2\,\delta_{\lambda_1\lambda_2}\delta(\mathbf{k}_1 - \mathbf{k}_2)$ which act very similarly to terms like $\left\langle \hat{a}_{\lambda_1}(\mathbf{k}_1)\,\hat{a}^{\dagger}_{\lambda_2}(\mathbf{k}_2)\right\rangle$ in the useful framework of QFT. This theory seems to recover the same experimentally verified results of QFT for phenomena such as the Casimir effect [4, 5] and the Unruh effect [25–27] for example, but eventually controversies rendered it less appealing, including recovering the ground state of the hydrogen atom [61] for example. Although it is interesting just how many quantum mechanical effects can be reproduced by the introduction of a random field.

In the situations considered in this thesis, the vacuum field is no longer necessarily Lorentz invariant as the symmetries of the system are disrupted by the presence of media. The author is unaware of any work attempting to consider how this random field would have to be adjusted to account for the effects of moving media.

I.4.2 The Wheeler-Feynman Absorber Theory

Another alternative view of the universe is the Wheeler-Feynman absorber theory [62, 63] which is briefly discussed in Feynman and Hibbs' book on path integrals [52]. The basis behind the theory lies in the realisation that the electric field as we know it was originally defined by the force that a positive test particle would experience if positioned at a certain point in space and time, with an analogous definition for the magnetic field. There seems to be no a priori reason why the electromagnetic field should be anything more than a mathematical shorthand as one never actually measures is directly, only its *effect*. The principle behind this theory was to therefore remove any references to the EM field and write all the dynamics of the particles in terms of all the other particles in the universe. In this sense, the Wheeler-Feynman absorber theory is an *action-at-a-distance* theory (since there is no direct "contact", or mediating field between distant interacting particles). In this view, the electromagnetic vacuum field would then merely represent a shorthand for the combination of the matter vacuum fields, i.e the fluctuations would come from the quantum fluctuations of the particles in the universe.

Another important idea in the Wheeler-Feynman absorber theory is the assumption of time-reversibility. This means that the absorption and emission of electromagnetic energy are actually the same effect, but time reversed. It also means that for every particle that emits energy, there must somewhere be a particle to absorb it otherwise reversing time would lead to particles absorbing energy from nothing. The result is that a particle in an otherwise empty universe could *never* emit radiation. Radiation can't be blindly catapulted out to oblivion.

There are curious conclusions to be drawn about the nature of the universe if such a theory were ever universally accepted. Firstly, predetermination would have to be solidly woven into the fabric of our universe. As an illustration of this point, consider the following example. This evening, wander outside and gaze up at the sky. What do you see? Stars. Light that has traversed the void for billions of years to fall upon the back of your eye after your seemingly unpredictable whim to wander into the night after dinner. Yet, according to this theory, light does not just radiate out with no given target, accidentally being absorbed along its path. This theory says that when that energy left the star, billions of years ago, it was destined to be absorbed by your eye on that evening, billions of years before your birth, or even the evolution of an 'eye'. There is something inherently beautiful about such a description, albeit easy fodder for sceptics and sensationalists alike. This predetermination goes against the currently accepted principles of quantum mechanics.

While this theory led to similar bursts of life for other action at a distance theories (see for example Hoyle and Narlikar's theory of gravity [64]), it has never been widely accepted since it couldn't correctly account for the self energy that leads to phenomena such as the Lamb shift [65–67] but remains an intriguing line of thought and an active area of research nonetheless.

I.5 Spontaneous Emission

To gain some insight into how the vacuum energy interacts with media, let's consider one of the most fundamental physical consequences of this theory, spontaneous emission. The simplest case is a two level system, initially in its excited state, which, due to the continuous fluctuations of the vacuum field, is eventually coerced into releasing its energy as a photon, de-exciting into its ground state.

The full system we consider here is a quantum harmonic oscillator of natural frequency ω_0 and dipole moment $\boldsymbol{\kappa}$ situated at a position $\mathbf{x}_0$ within the electromagnetic vacuum described in the previous section. The coupling between them is treated as a perturbation to the uncoupled system. The full Lagrangian of the system is therefore defined in terms of the free electromagnetic Lagrangian L_0, the oscillator Lagrangian L_ω of (I.26) and the oscillator amplitude X as

$$L = L_0 + L_\omega + \int d^3\mathbf{x}\,\mathsf{X}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\boldsymbol{\kappa}\cdot\mathbf{E}. \tag{I.54}$$

Considering the problem using the full canonical formalism from QFT; the momentum of the field is found via to (I.36), but now contains an extra term from the coupling

$$\Pi_{\mathbf{A}} = \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{A}}} = -\epsilon_0\mathbf{E} - \mathsf{X}\,\boldsymbol{\kappa}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) \tag{I.55}$$

and the momentum of the oscillator is given by (I.27). The Hamiltonian is found similarly to (I.28) and (I.43) and can be written in terms of a part that depends on the dipole moment H_I and a part independent of the dipole moment, H_0 where the full Hamiltonian is given by $\hat{H} = \hat{H}_0 + \hat{H}_I$. We work in the interaction picture [53, 68], where the time dependence of the operators is generated by the bare

Hamiltonian $\hat{H}_0$, given by the operator equivalent of H_0 and that of the quantum state by the interaction Hamiltonian $\hat{H}_I(t) = e^{-i\hat{H}_0 t}\hat{H}_I\, e^{i\hat{H}_0 t}$, the operator equivalent of H_I. The interaction part is given to first order in $\boldsymbol{\kappa}$ by

$$\hat{H}_I\left(t\right) = \int d^3\mathbf{x}\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_0\right) \hat{\mathrm{X}}\left(t\right) \boldsymbol{\kappa} \cdot \hat{\mathbf{E}}\left(\mathbf{x}, t\right). \tag{I.56}$$

Now consider any system whose wave function satisfies the time dependant Schroedinger equation

$$\hat{H}\left(t\right)\left|\psi\left(t\right)\right\rangle = i\hbar\partial_t \left|\psi\left(t\right)\right\rangle \tag{I.57}$$

where $\left|\psi\left(t\right)\right\rangle$ represents the state of the system in the Schroedinger picture and the time dependent Hamiltonian $H\left(t\right)$ can be written as some time independent term H_0 and a perturbation term $H_S\left(t\right)$ which is turned on at some initial $t = t_0$, where H_0 is diagonalised in terms of some set of creation and annihilation operators. Here, these will be the free field operators of (I.42) and the quantum harmonic oscillator operators of (I.30). Transforming the wave function $\left|\psi\left(t\right)\right\rangle$ to the interaction picture via the unitary transformation

$$\left|\psi\left(t\right)\right\rangle = e^{-i\hat{H}_0 t/\hbar}\left|\psi_I\left(t\right)\right\rangle, \tag{I.58}$$

and substituting into (I.57) leads to the following equation for the interaction wave function

$$\hat{H}_I\left(t\right)\left|\psi_I\left(t\right)\right\rangle = i\hbar\partial_t \left|\psi_I\left(t\right)\right\rangle, \tag{I.59}$$

where we have defined the interaction Hamiltonian via the unitary transformation as

$$\hat{H}_I\left(t\right) = e^{i\hat{H}_0 t/\hbar}\hat{H}_S\left(t\right) e^{-i\hat{H}_0 t/\hbar}. \tag{I.60}$$

After integration over time, this leads to the solution

$$\left|\psi_I\left(t\right)\right\rangle = \left|\psi_I\left(t_0\right)\right\rangle - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\ \hat{H}_I\left(t\right)\left|\psi_I\left(t_1\right)\right\rangle. \tag{I.61}$$

Iteratively substituting this solution back into itself gives the complete solution as a series of integrals known as the Dyson series given by

$$\begin{aligned}\left|\psi_I\left(t\right)\right\rangle &= \left|\psi_I\left(t_0\right)\right\rangle + \left(-\frac{i}{\hbar}\right)\int\limits_{t_0}^{t} dt_1\ \hat{H}_I\left(t_1\right)\left|\psi_I\left(t_0\right)\right\rangle \\ &\quad + \left(-\frac{i}{\hbar}\right)^2\int\limits_{t_0}^{t} dt_1\ \hat{H}_I\left(t_1\right)\int\limits_{t_0}^{t_1} dt_2\ \hat{H}_I\left(t_2\right)\left|\psi_I\left(t_0\right)\right\rangle + \text{...etc...}\end{aligned} \tag{I.62}$$

If the measure of the interaction Hamiltonian is small enough, which is true for this weak coupling approach, higher order terms may be discarded. Let us consider only the first order term and approximate the interaction wave function as

$$\left|\psi_I\left(t\right)\right\rangle \approx \left|\psi_I\left(t_0\right)\right\rangle - \frac{i}{\hbar}\int\limits_{t_0}^{t} dt_1\ \hat{H}_I\left(t_1\right)\left|\psi_I\left(t_0\right)\right\rangle \tag{I.63}$$

and returning to the Schroedinger picture gives

$$\left|\psi\left(t\right)\right\rangle \approx e^{-i\hat{H}_0 t/\hbar}\left|\psi_I\left(t_0\right)\right\rangle - \frac{i}{\hbar}e^{-i\hat{H}_0 t/\hbar}\int\limits_{t_0}^{t} dt_1\ \hat{H}_I\left(t\right)\left|\psi_I\left(t_0\right)\right\rangle . \tag{I.64}$$

By considering the electromagnetic field to be initially in its ground state and the two-level system to be in its excited state, we can write the initial state of the system as $\left|\psi\left(0\right)\right\rangle = \left|1\right\rangle_{\mathsf{X}} \otimes \left|0\right\rangle_{\mathsf{EM}}$. After some time t, the atom will emit a photon and descend into its ground state leaving the electromagnetic field containing one photon. One therefore finds that the wave function for the system becomes

$$\left|\psi\left(t\right)\right\rangle = \left|1\right\rangle_{\mathsf{X}} \otimes \left|0\right\rangle_{\mathsf{EM}} + \int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\sum_{\lambda}\xi_{\lambda}\left(\mathbf{k},t\right)\left|0\right\rangle_{\mathsf{X}} \otimes \left|1_{\mathbf{k}}^{\lambda}\right\rangle_{\mathsf{EM}} \tag{I.65}$$

where the rate of change of the quantity ξ is found by taking the derivative of (I.65) with respect to time and remembering that the unperturbed states of the system the Schroedinger equation (I.57), i.e

$$\begin{aligned}\hat{H}_0\left|\psi_0\right\rangle &= i\hbar\partial_t\left|\psi_0\right\rangle \\ \hat{H}_0\left|0\right\rangle_{\mathsf{X}} \otimes \left|1_{\mathbf{k}}^{\lambda}\right\rangle_{\mathsf{EM}} &= i\hbar\partial_t\left(\left|0\right\rangle_{\mathsf{X}} \otimes \left|1_{\mathbf{k}}^{\lambda}\right\rangle_{\mathsf{EM}}\right)\end{aligned} \tag{I.66}$$

leads us to the expression

$$\hat{H}_I(t)\left[|1\rangle_{\mathsf{X}}\otimes|0\rangle_{\mathsf{EM}}+\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda}\xi_\lambda(\mathbf{k},t)\,|0\rangle_{\mathsf{X}}\otimes\left|1^\lambda_{\mathbf{k}}\right\rangle_{\mathsf{EM}}\right]=\\ i\hbar\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda}\dot{\xi}_\lambda(\mathbf{k},t)\,|0\rangle_{\mathsf{X}}\otimes\left|1^\lambda_{\mathbf{k}}\right\rangle_{\mathsf{EM}}. \tag{I.67}$$

Applying the operator $\left\langle 1^{\lambda'}_{\mathbf{k}'}\right|_{\mathsf{EM}}\otimes\langle 0|_{\mathsf{X}}$ from the left and simplifying, we arrive at

$$\dot{\xi}_\lambda(\mathbf{k},t)=-\frac{i}{\hbar}\left\langle 1^\lambda_{\mathbf{k}}\right|_{\mathsf{EM}}\otimes\langle 0|_{\mathsf{X}}\;\hat{H}_I(t)\,|1\rangle_{\mathsf{X}}\otimes|0\rangle_{\mathsf{EM}} \tag{I.68}$$

where the second term of (I.67) disappears due to the interaction Hamiltonian being quadratic. Note that since we are only considering the first order in perturbation theory for a quadratic interaction Hamiltonian, the wave function (I.65) is already normalised[4]. The interaction Hamiltonian is given by (I.56). For our purposes, due to the annihilation of the vacuum, this Hamiltonian operator is equivalent to

$$\hat{H}_I(t)\equiv-\frac{i\hbar}{2}\sum_{\lambda}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{\frac{\omega(\mathbf{k})}{\omega_0}}\boldsymbol{\kappa}\cdot\mathbf{e}^\lambda(\mathbf{k})\,e^{-i[\mathbf{k}\cdot\mathbf{x}_0-(\omega(\mathbf{k})-\omega_0)t]}\,\hat{b}\,\hat{a}^\dagger_\lambda(\mathbf{k}) \tag{I.69}$$

where we have used the fact $\hat{b}\,|0\rangle_{\mathsf{X}}=\hat{a}_\lambda\,|0\rangle_{\mathsf{EM}}=0$ (which leads to the three extra terms disappearing). The rate of change of the quantity ξ can then be found to be

$$\dot{\xi}_\lambda(\mathbf{k},t)=-\frac{1}{2}\sqrt{\frac{\omega(\mathbf{k})}{\omega_0}}\boldsymbol{\kappa}\cdot\mathbf{e}^\lambda(\mathbf{k})\,e^{i[\mathbf{k}\cdot\mathbf{x}_0-(\omega(\mathbf{k})-\omega_0)t]}. \tag{I.70}$$

We wish to find the average emission rate for long times therefore we integrate from $-T/2$ to $T/2$ yielding

$$\xi_\lambda(\mathbf{k},T)=-\sqrt{\frac{\omega(\mathbf{k})}{\omega_0}}\boldsymbol{\kappa}\cdot\mathbf{e}^\lambda(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}_0}\frac{\sin\left[(\omega_0-\omega(\mathbf{k}))\,T/2\right]}{(\omega_0-\omega(\mathbf{k}))}. \tag{I.71}$$

Taking the absolute value and squaring, then dividing by T and taking the limit

[4]See for example [49]

$T \to \infty$ gives

$$\lim_{T\to\infty} \frac{\left|\xi_\lambda\left(\mathbf{k}, T\right)\right|^2}{T} = \frac{\omega\left(\mathbf{k}\right)}{\omega_0} \left|\boldsymbol{\kappa} \cdot \mathbf{e}^\lambda\left(\mathbf{k}\right)\right|^2 \lim_{T\to\infty} \frac{\left|\sin\left[\left(\omega_0 - \omega\left(\mathbf{k}\right)\right)T/2\right]\right|^2}{\left|\omega_0 - \omega\left(\mathbf{k}\right)\right|^2 T}. \tag{I.72}$$

This yields

$$\lim_{T\to\infty} \frac{\left|\xi_\lambda\left(\mathbf{k}, T\right)\right|^2}{T} = \frac{\pi\omega\left(\mathbf{k}\right)}{2\omega_0} \left|\boldsymbol{\kappa} \cdot \mathbf{e}^\lambda\left(\mathbf{k}\right)\right|^2 \delta\left[\omega_0 - \omega\left(\mathbf{k}\right)\right] \tag{I.73}$$

where we have applied the result

$$\lim_{T\to\infty} \frac{4\pi\left|\sin\left[\left(\omega_0 - \omega\left(\mathbf{k}\right)\right)T/2\right]\right|^2}{\left|\omega_0 - \omega\left(\mathbf{k}\right)\right|^2 T} = 2\pi\delta\left[\omega_0 - \omega\left(\mathbf{k}\right)\right]. \tag{I.74}$$

If we use the property that $\delta[f(x)] = \delta(x - x_0)/f'(x_0)$, integrating over all wave-vectors yields the rate of spontaneous emission

$$\Gamma_{1\to0} = \frac{1}{16\pi^2} \sum_\lambda \int dk \int d\Omega \, \frac{k^3}{\omega_0} \left|\boldsymbol{\kappa} \cdot \mathbf{e}^\lambda\left(\mathbf{k}\right)\right|^2 \delta\left(k - \frac{\omega_0}{c}\right) \tag{I.75}$$

where the angle Ω is defined to be the angle between the wave-vector $\mathbf{k}$ and the dipole direction $\boldsymbol{\kappa}$. From this definition we can also write $\sum_\lambda \left|\boldsymbol{\kappa} \cdot \mathbf{e}^\lambda\left(\mathbf{k}\right)\right|^2 = |\boldsymbol{\kappa}|^2 \sin^2\theta$. Performing the integrals reveals the spontaneous emission rate

$$\Gamma_{1\to0} = \frac{2\omega_0^2}{3\pi c^3} \left|\boldsymbol{\kappa}\right|^2. \tag{I.76}$$

Notice that there is a quadratic dependence of the emission rate on the frequency of the oscillator, so the higher the energy level, the more likely the two level system is to emit a photon. Often this result is quoted as having a cubic dependence, however the $1/\omega$ dependence of the dipole amplitude results in the same overall dependence. This simple calculation illustrates that the vacuum field that exists throughout space interacts with particles in a significant manner.

The purpose of this work is to understand how the vacuum field is changed by the presence of moving or time-dependent media. It therefore seems natural to consider a very similar situation within moving media and ask ourselves, how does moving media effect the evolution of the quantum vacuum?

Chapter II

The Quantum Vacuum Near Moving Media

In trying to understand the mechanisms underpinning any phenomenon, it is wise to try to consider the simplest of cases first. Therefore we begin by considering the case of a single two-level system travelling at constant velocity through a homogeneous material of constant, *real* refractive index $n \in \mathbb{R}$. The lack of a complex part of the refractive index results in a medium that does not absorb electromagnetic energy. This simple system also has the added bonus of resembling the spontaneous emission case previously treated, and the two level system may then be interpreted as a probe of the quantum vacuum. The purpose of this chapter is to use the dynamics of this probe as it travels through the media to further understand the effects of the relative motion on the quantum vacuum.

The first two sections of this chapter are included in the findings of the paper [1]. The present author's contribution to [1] was primarily the quantum mechanical model, while the first author of [1] was primarily responsible for the classical case. The third section of this chapter, while unpublished, is the supervised work of the current author.

II.1 Homogeneous Media

For now, let us consider the physics of a two level system, modelled using a harmonic oscillator, being dragged through a homogeneous, non-absorbing dielectric. As mentioned in the introduction, this is similar to the setup in which one observes Cherenkov radiation [13], the main difference being that our two-level system is electrically neutral but is polarised by the quantum vacuum. For simplicity, we work in $1+1$ dimensions and use a simple scalar field to represent the electromagnetic field. First we treat the classical case, then the quantum mechanical case. This scalar field is essentially the first spatial part of EM potential field $\mathbf{A}$ described in section I.2. After making the replacement $\mathbf{x} \to x$, one also replaces the potential field by its x component A_x which we now write as φ. According to (I.1), this means also making the replacement $\mathbf{E} \to \partial_t \varphi$.

It is important to point out that there are many flaws in this simplified model: in reality the material is not completely homogeneous and this fact becomes important in the vicinity of the atom which, in reality, would be directly surrounded by a vacuum. Unlike in [69, 70], we ignore this empty space surrounding this harmonic oscillator which should only become important for a specific range of oscillator frequencies. We also begin by ignoring dissipation of energy. We stress that, even in this case where the dipole is travelling through a medium, none of the effects derived are due to collisions per se, but simply the interaction with fluctuating electromagnetic ground state energy. We completely ignore how this harmonic oscillator manages to travel through this material without collisions.

Nevertheless, this model is sufficient to explore some of the main effects underpinning the interaction between moving media as mediated by the vacuum field. Note that the classical case is derived in the frame of the oscillator, and the quantum mechanical case, in the frame of the medium. Special relativity requires that the physics must be independent of reference frame, and one can of course show this by transforming the final results from one frame to the other using the Lorentz transformation (I.2).

II.1.1 Classical Radiation Damping

The Lagrangian density for a scalar field within a medium with a homogeneous, constant, real refractive index n is similar to the scalar equivalent of the vacuum Lagrangian (I.34) except for the appearance of the refractive index,

$$\mathscr{L}_0 = -\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{n^2}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2. \tag{II.1}$$

According to the Euler-Lagrange equations (I.35), this scalar field obeys the wave equation,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi(x,t) = 0. \tag{II.2}$$

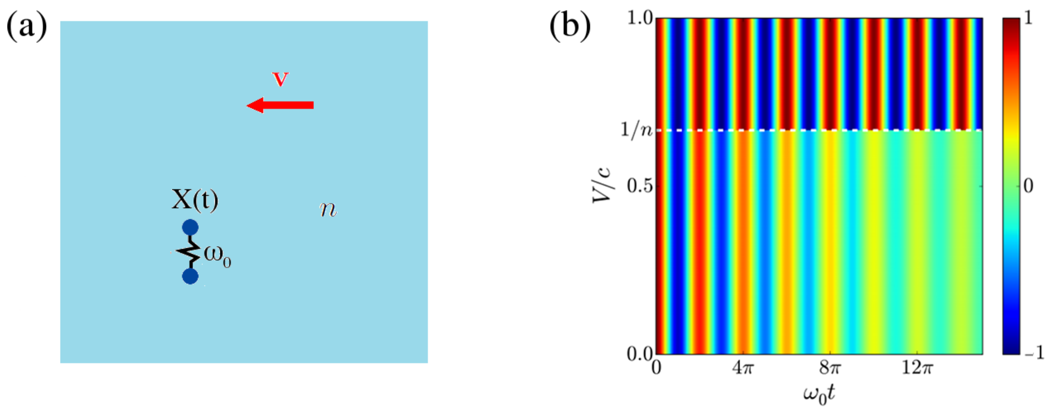

Figure II.1: (a) In the frame where the medium is in motion a small test particle is imagined to be embedded at a fixed position x_0. This particle has an internal degree of freedom X(t) that behaves as a harmonic oscillator, coupled to the field φ. When the medium is at rest the coupling to the field serves to damp the internal degree of freedom. Meanwhile when the medium is moving, those frequencies that have changed sign between the rest frame and the moving frame serve to amplify the motion of the internal degree of freedom. (b) Plot of the motion of X(t) as a function of time and velocity computed using (II.14) though a medium with refractive index n (initial conditions: X$(0) = 1$ and $\dot{\text{X}}(0) = 0$). Below the critical velocity $v = c/n$, the oscillation is damped by the coupling to the field, a damping which disappears for velocities $v > c/\mathrm{n}$. The disappearance of the damping is due to a cancellation between the energy gained/lost from the k_+ and k_- modes, which both propagate left when $v > c/n$.

If we now change frame of reference to that in which the medium is in motion with velocity $-v$, the coordinates transform according to the one dimensional equivalent of (I.2) and the derivatives according to the one dimensional equivalent of (I.5) after having made the change $v \to -v$ in both. In other words, we use the transformations

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &\to \lambda^2\left(\frac{\partial}{\partial x'}-\frac{v}{c^2}\frac{\partial}{\partial t'}\right)^2\\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2} &\to \lambda^2\left(\beta\frac{\partial}{\partial x'}-\frac{1}{c}\frac{\partial}{\partial t'}\right)^2.\end{aligned} \tag{II.3}$$

The scalar field is then coupled to a harmonic oscillator with coupling strength κ and our wave equation becomes

$$\begin{aligned}\lambda^2\bigg[\left(1-n^2\beta^2\right)\frac{\partial^2}{\partial x'^2}-\left(n^2-\beta^2\right)\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}+2\frac{(n^2-1)\,\beta}{c}\frac{\partial}{\partial t'}\frac{\partial}{\partial x'}\bigg]\varphi(x',t')\\ =-\kappa\delta(x'-x'_0)\dot{\mathsf{X}}(t').\end{aligned} \tag{II.4}$$

If we 'add zero', we can rewrite this in the following form

$$\begin{aligned}\lambda^2\bigg[\left(1-\beta^2+\beta^2-n^2\beta^2\right)\frac{\partial^2}{\partial x'^2}-\left(n^2-1+1-\beta^2\right)\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}+2\frac{(n^2-1)\,\beta}{c}\frac{\partial}{\partial t'}\frac{\partial}{\partial x'}\bigg]\\ \times\varphi(x',t')=-\kappa\delta(x'-x'_0)\dot{\mathsf{X}}(t').\end{aligned} \tag{II.5}$$

Now if we recombine the terms using the fact that $\lambda^2(1-\beta^2)=1$, the wave equation can be written in the form

$$\left[\frac{\partial^2}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}-\frac{\gamma^2(\mathrm{n}^2-1)}{c^2}\left(\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right)^2\right]\varphi(x',t')=-\kappa\delta(x'-x'_0)\dot{\mathsf{X}}(t'). \tag{II.6}$$

Note that the LHS is no longer the Lorentz invariant wave equation of the vacuum since the presence of the refractive index n breaks the invariance. However, as n$\to 1$, the invariance of the vacuum is recovered as the third term vanishes. The harmonic oscillator obeys the equation

$$\ddot{\mathsf{X}}\left(t'\right)+\omega_0^2\mathsf{X}\left(t'\right)=-\kappa\dot{\varphi}\left(x'_0,t'\right). \tag{II.7}$$

To find the solutions to equation (II.6) and (II.8), we choose to expand the amplitudes in terms of their Fourier components via

$$\mathsf{X}\left(t'\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathsf{X}\left(\omega\right) e^{-i\omega t'} \tag{II.8}$$

and

$$\varphi\left(x', t'\right) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \varphi\left(k, \omega\right) e^{i\left(kx' - \omega t'\right)}. \tag{II.9}$$

Fourier transforming (II.6) gives us the Fourier components in (II.9) in terms of the Fourier components of (II.8), i.e

$$\varphi(x', t') = -\frac{i\kappa}{\gamma^2(1 - n^2\beta^2)} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\left[k(x' - x_0') - \omega t'\right]}}{\left(k - \mathrm{k}_-\right)\left(k - \mathrm{k}_+\right)} \omega \tilde{X}(\omega) \tag{II.10}$$

where

$$\mathrm{k}_\pm = \frac{\omega + \mathrm{i}\eta}{c} \left[\frac{(n^2 - 1)\beta \pm n/\gamma^2}{1 - n^2\beta^2}\right]. \tag{II.11}$$

The two roots $k_\pm$ occur where the dispersion relation for the waves in the moving frame is fulfilled. These are plotted in the complex plane in Figure II.2. The quantity η is an infinitesimal positive constant that serves to shift the $k_\pm$ into the lower half complex frequency plane which amounts to enforcing causality. For a more detailed discussion of this shift $\mathrm{i}\eta$, see the Appendix A.1. Inserting this expression into (II.7) and then taking a Fourier transform gives an equation for the Fourier components $\tilde{\mathsf{X}}(\omega)$ of (II.8) that requires ω to satisfy the following relation

$$\omega^2 - \omega_0^2 + \frac{\kappa^2\omega^2}{\gamma^2(1 - n^2\beta^2)} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\left(k - \mathrm{k}_-\right)\left(k - \mathrm{k}_+\right)} = 0. \tag{II.12}$$

As a reminder, the equivalent equation for a damped harmonic oscillator is of the form $\omega^2 - \omega_0^2 + i\Gamma = 0$ where the $i\Gamma$ term is responsible for the damping. As such, the imaginary part of the integral in (II.12) determines the extent to which the coupling to the field has a damping effect on the motion of the oscillator,

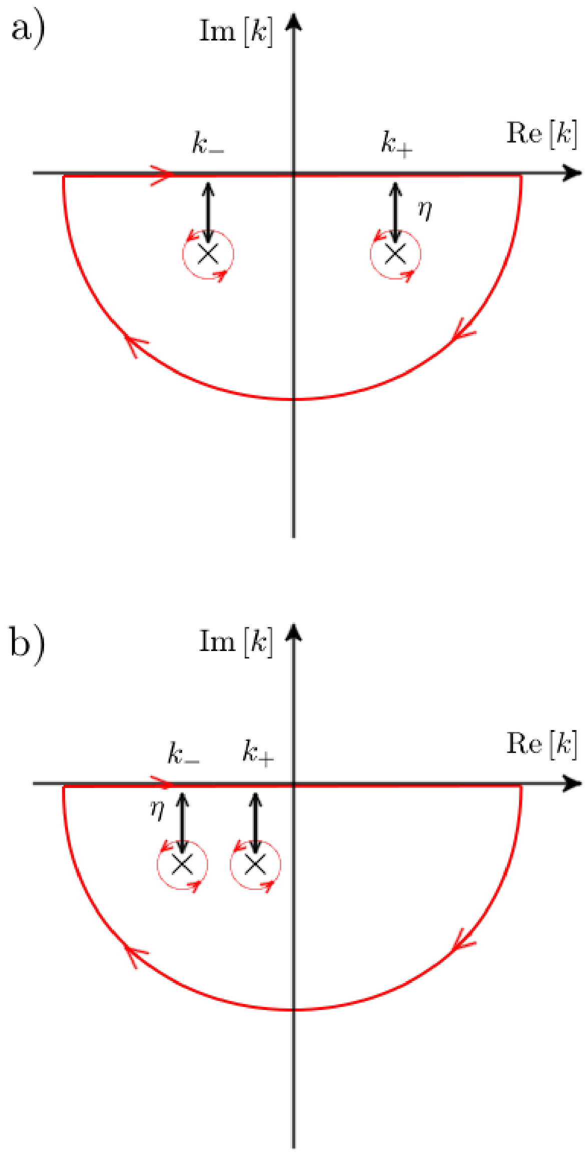

Figure II.2: (a) Analytic continuation of the integral over the wave–vector k in expression (II.13) results in the red contour. The small imaginary part $i\eta$ in the poles $k_\pm$ serves to shift them off the real axis into the lower half plane. (b) As the velocity v increases the poles move position in the complex plane. For speeds that exceed the velocity of light in the medium, i.e when $v > c/n$, the positive pole k_+ changes sign and poles pole appear in the left hand side of the imaginary axis. When this occurs, the contribution from the poles cancel each other out and the oscillator damping by the field disappears.

equivalent to a damping constant

$$\Gamma = \frac{\kappa^2\omega^2}{\gamma^2(1-n^2\beta^2)}\text{Im}\left[\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{(k-\mathrm{k}_-)\,(k-\mathrm{k}_+)}\right]. \tag{II.13}$$

The damping has contributions from both of the roots $k_\pm$ and can be isolated through taking the limit $\eta \to 0$ and deforming the contour of integration to skirt around the poles on the real k axis

$$\Gamma = \frac{c\kappa^2}{4\mathrm{n}}\left[\mathrm{sign}(\mathrm{k}_+) - \mathrm{sign}(\mathrm{k}_-)\right] \tag{II.14}$$

For $|v| < c/\mathrm{n}$ (when the frequencies have the same sign in the moving frame and the rest frame of the medium), k_+ and k_- have opposite signs and (II.14) is positive, meaning that the motion of the oscillator is damped by its coupling to the field. Evidently in this regime both solutions k_+ and k_- contribute equally to this damping. Meanwhile, when $|v| > c/\mathrm{n}$, k_+ and k_- have the same sign and (II.14) vanishes, the coupling to the field no longer providing any damping of the oscillator. This is because one of the two modes now amplifies the oscillatory motion with an equal magnitude to the damping due to the second mode. The amplifying mode has a frequency of a different sign in the moving frame compared to the rest frame of the medium, due to the fact that $|v| > c/\mathrm{n}$. This behaviour of X(t) is shown in figure II.1b.

In this simple model, it is clear that the change in the sign of the frequency is a crucial factor in the amplification of the oscillator amplitude as it travels through the medium. It is also clear that the inclusion of a macroscopic description of a dielectric medium through a refractive index $\mathrm{n} > 1$ is fundamental to the change of sign of the frequency (the effect disappears when $\mathrm{n} = 1$ since the velocity of the medium would have to exceed c).

A question one might ask now is what is the meaning of this change of sign of the frequency? To shed some light on the matter, consider a plane wave φ moving with a phase velocity $v_p = \omega/k$; this can be written as,

$$\varphi(x,t) = \cos(kx \pm \omega(k)t) = \cos[k(x \pm v_p t)] \tag{II.15}$$

where the frequency $\omega(k)$ is a function of wave-vector k. All that happens when $\omega \to -\omega$ is that the wave reverses its direction, $v_p \to -v_p$. In this respect, the relative sign of ω and k is all that matters, as it encodes the propagation direction of the wave. Typically the convention is that ω is positive so that the direction is determined by the sign of k. Yet in some situations, the change in sign occurs due to a change of reference frame. Here, the waves appear to reverse their direction through a change in sign of the frequency rather than wave–vector.

Now consider an example to illustrate when we should expect frequencies to change sign between reference frames. Imagine two cases where an observer travels relative to an electromagnetic wave: (i) moving through free space at velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ relative to a light wave with a wave vector at angle θ, $\boldsymbol{k} = (\omega/c)[\cos(\theta)\hat{\boldsymbol{x}} + \sin(\theta)\hat{\boldsymbol{y}}]$; and (ii) moving through a medium at the same velocity but parallel to a light wave $\boldsymbol{k} = (\mathrm{n}\omega/c)\hat{\boldsymbol{x}}$, where $\mathrm{n} > 1$ is the refractive index of the medium. In both cases the moving observer sees a wave with wave–vector and frequency that have undergone a Lorentz transformation [46]

$$\begin{aligned} k'_x &= \gamma\left(k_x - \frac{\omega v}{c^2}\right) \\ \omega' &= \gamma\left(\omega - v k_x\right) \end{aligned} \tag{II.16}$$

where, again, $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = v/c$. In both cases (i) and (ii) the observer can travel fast enough for the wave to appear to reverse its direction of propagation. In case (i) this occurs when $v > c\cos(\theta)$, which is where k'_x takes a different sign in the frame of the moving observer, with ω' remaining positive. Yet in case (ii) the wave appears to reverse direction when $v > c/\mathrm{n}$ where the transformed wave vector k'_x remains positive, but the frequency ω' changes sign. Consequently, even though we conventionally take all frequencies to be positive in one frame of reference, changing reference frame can cause some of these frequencies to change sign. Applying the formula $E = \hbar\omega'$, one is led to the conclusion that positive energy quanta in the rest frame appears as negative energy quanta in the moving frame (since ω' becomes negative in such a frame). Such negative energy quanta imply a kind of instability where waves can be excited by a moving particle, while lowering the energy of the system. Such waves, which from now on will be referred to as *negative frequency waves*, were recognised during the early work on quantum electrodynamics by authors such as Jauch and

Watson [71, 72], but still remain somewhat mysterious. A recent more detailed analysis that includes the dispersion and dissipation of the medium [44] shows that these modes cause the Hamiltonian of the field to lack a lower bound, and are the origin of the force of quantum friction [73].

II.1.2 Quantum Mechanical Absorption Rate

In the previous section, we derived the classical damping experienced by an oscillator as it travels through a homogeneous dielectric with a real refractive index. This damping disappears when the frequency of certain modes of light change sign compared to their rest frame values, which occurs when the velocity of the medium exceeds the velocity of light within this medium. The damping or amplification of the oscillator amplitude, of course, relies on the oscillator having a non-zero oscillation amplitude to begin with. It was mentioned in section I.5 that the quantum vacuum is constantly fluctuating (this was also the basis behind the stochastic electrodynamics theory of section I.4.1.) This means that at the quantum scale, one can think of the oscillator as being perpetually kept in motion by this random background field. Our question is now, if one supposes that the oscillator of the previous section was subject to *only* these quantum fluctuations, what would be the likelihood of its absorbing enough energy from the vacuum to move into its excited state?

The frame of reference is taken to the that of the medium and not the oscillator. The position of the oscillator travels along with a velocity v with respect to the medium. Setting $x_0 = 0$, its position is therefore vt. The Lagrangian density is then

$$\mathscr{L} = \mathscr{L}_0 - \kappa \mathsf{X} \delta(x - vt) \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} \right) + \mathscr{L}_{Osc} \tag{II.17}$$

where $\mathscr{L}_{Osc}$ is the Lagrangian density for the free oscillator. In the frame of reference in which the oscillator is at rest, this would be given by (I.25). In the frame of the medium, the oscillator is now motion and the frequency ω_0 of the oscillator thus becomes the scaled frequency ω_0/γ. Ensuring that the action $S = \int L \, dt$ is Lorentz invariant leads to a Lorentz factor γ appearing in the Lagrangian which we now write as

$$L_{Osc} = \frac{\gamma}{2}\left[\dot{\mathsf{X}}^2 - \left(\frac{\omega_0}{\gamma}\right)^2 \mathsf{X}^2\right]. \tag{II.18}$$

The full Lagrangian density is thus given by

$$\mathscr{L} = \mathscr{L}_0 - \kappa \mathsf{X}\delta(x - vt)\left(\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x}\right) + \frac{\gamma}{2}\left[\dot{\mathsf{X}}^2 - \left(\frac{\omega_0}{\gamma}\right)^2 \mathsf{X}^2\right]\delta(x - vt). \tag{II.19}$$

In order to quantise the system, we can again use the Euler-Lagrange equations (I.35) to (II.19), to find the canonical momentum of the field

$$\Pi_\varphi = \frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)} = \frac{n^2}{c^2}\frac{\partial\varphi}{\partial t} - \kappa\mathsf{X}\delta(x - vt) \tag{II.20}$$

and that of the oscillator by

$$p_{\mathsf{X}} = \frac{\partial L}{\partial(\partial_t \mathsf{X})} = \gamma\frac{\partial \mathsf{X}}{\partial t} \tag{II.21}$$

from which one can construct the Hamiltonian of the system. This can then be written in terms of these canonical variables via

$$\begin{aligned} H &= \int_{-\infty}^{\infty} dx\, \Pi_\varphi \cdot \frac{\partial\varphi}{\partial t} + p_{\mathsf{X}} \cdot \frac{\partial X}{\partial t} - L \\ &= \int_{-\infty}^{\infty} dx \left\{\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{c^2}{2n^2}\left[\Pi_\varphi + \kappa X\delta\left(x - vt\right)\right]^2 + \kappa\mathsf{X}\delta\left(x - vt\right)v\frac{\partial\varphi}{\partial x}\right\} \\ &+ \frac{1}{2\gamma}\left(p_{\mathsf{X}}^2 + \omega_0^2\mathsf{X}^2\right). \end{aligned} \tag{II.22}$$

Similarly to section (I.5), one can once again write this Hamiltonian as sum of a time independent part H_0 and a time dependent part, the *interaction* Hamiltonian $H_I(t)$, i.e. $H = H_0 + H_I(t)$. Since the two-level system is now travelling within the medium, the interaction Hamiltonian operator in this case turns out to be different and, to first order in κ, it is found to be

$$\hat{H}_I = \kappa\int_{-\infty}^{\infty} dx\hat{\mathsf{X}}\delta\left(x - vt\right)\left[\frac{c^2}{n^2}\hat{\Pi}_\varphi + v\frac{\partial\hat{\varphi}}{\partial x}\right]. \tag{II.23}$$

Promoting the canonical fields amplitudes to operators concludes the second quantisation where the expansion of the field operators are written in terms of a set of one dimensional creation and annihilation operators that diagonalise the stationary Hamiltonian operators $\hat{H}_0$. These are given as

$$\hat{\varphi}(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\hbar c^2}{2n^2\omega(k)}} \left[\hat{a}(k) e^{i(kx-\omega(k)t)} + \text{h.c}\right] \tag{II.24}$$

with the corresponding momentum give as

$$\hat{\Pi}_{\varphi}(x,t) = -i \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\hbar c^2}{2n^2\omega(k)}} \left[\hat{a}(k) e^{i(kx-\omega(k)t)} - \text{h.c}\right]. \tag{II.25}$$

where the creation and annihilation operators are analogous to those of (I.42) and obey

$$\left[\hat{a}\left(k_1\right), \hat{a}^{\dagger}\left(k_2\right)\right] = 2\pi\delta\left(k_1 - k_2\right). \tag{II.26}$$

Note the appearance of the refractive index in the denominator of the normalisation coefficients in (II.24) and (II.25), in the limit of a transparent medium where $n^2 \to \epsilon_0$, these recover the one dimensional scalar analogues of (I.39).

In the spontaneous emission calculation, the electromagnetic field was initially considered to be in its ground state and the two-level system in its excited state. In the final state of the system, the EM field contained one photon and the dipole was then in its ground state. Here, the situation is different. We consider both the the scalar field and the two-level system to be in their ground state initially. The final state is when the dipole is excited to its first level and the field is also excited. Similarly to the classical situation described in the last section, this corresponds to a negative quanta of energy being absorbed by the field with its positive counterpart exciting the dipole. In a very similar fashion to section I.5, the system is found to be described by the following wave function

$$\begin{aligned} |\psi\rangle &= |0\rangle_{\mathsf{X}} \otimes |0\rangle_{\phi} + \int \frac{dk}{2\pi} \xi(k,t) \, |1\rangle_{\mathsf{X}} \otimes |1(k)\rangle_{\phi} \\ \langle\psi| &= \langle 0|_{\phi} \otimes \langle 0|_{\mathsf{X}} + \int \frac{dk}{2\pi} \xi(k,t) \, \langle 1(k)|_{\phi} \otimes \langle 1|_{\mathsf{X}} \end{aligned} \tag{II.27}$$

where the rate of change of the quantity ξ is now given by

$$\dot{\xi}(k,t) = -\frac{i}{\hbar}\left\langle 1(k)\right|_{\phi} \otimes \left\langle 1\right|_{\mathsf{X}} H_I(t)\left|0\right\rangle_{\mathsf{X}} \otimes \left|0\right\rangle_{\phi}. \tag{II.28}$$

Inserting the expression (II.23) into (II.28), and using expressions (II.24) and (II.25) yields an expression for the rate of change of the quantity ξ

$$\dot{\xi}(k,t) = \mathrm{i}\frac{\kappa c}{2n}\frac{\left(-\omega(k)+vk\right)}{\sqrt{\omega_0\omega(k)}}e^{-i\left(kv-\omega(k)-\frac{\omega_0}{\gamma}\right)t}. \tag{II.29}$$

As before, integrating over some time T, taking the absolute then the limit as $T \to \infty$ then integrating over all the modes of the system, we find that the absorption rate of the moving dipole within the medium is

$$\Gamma_{1\to 0} = \frac{\kappa^2 c^2}{4n^2}\int_{-\infty}^{\infty} dk\,\frac{\left(\omega(k)-vk\right)^2}{\omega_0\omega(k)}\delta\left(kv-\omega(k)-\frac{\omega_0}{\gamma}\right) \tag{II.30}$$

where we have used the identity (I.74). Using the property $\delta[f(x)] = \delta(x - x_0)/f'(x_0)$, this can be rewritten as

$$\Gamma_{1\to 0} = \frac{\kappa^2 c^2}{4n^2}\int_{-\infty}^{\infty} dk\frac{\left(\omega(k)-vk\right)^2}{\omega_0\omega(k)\left(v-\partial\omega(k)/\partial k\right)}\delta\left(k-k_0\right) \tag{II.31}$$

where the wave-vector k_0 is given by

$$k_0 = \frac{\omega_0}{\gamma\left(v - c\,\mathsf{sign}\left(k\right)n\right)}. \tag{II.32}$$

Finally, integrating over all values of wave-vector k yields the total emission rate

$$\Gamma_{1\to 0} = \frac{\kappa^2 c}{4n\gamma}\theta\left(\frac{nv}{c}-1\right). \tag{II.33}$$

This expressions tells us that when the dipole is at rest in the medium, both it and the scalar field remain in their grounds states. For low velocities, the result is the same. However, when the velocity exceeds c/n, i.e the speed of light in the medium, there is a finite probability that the dipole will transition to its excited state. This result is closely analogous to the disappearance of the damping in the classical case when the velocity exceeds c/n where the oscillator amplitude is amplified. The quantum of energy required for two-level system to excite comes from the scalar field. It is thought that this quanta of energy will in fact come from the kinetic energy of the centre of mass of the medium, via the coupling to the scalar field. This will result in a reduction in the kinetic energy of the medium by that very same energy thus there is a strong connection within this effect to the 'quantum friction' touched upon in the introductory chapter.

Furthermore, note that in this case, the particle is inside the medium, but would the same result occur if the particle were outside the medium, near the surface for example? We've also omitted the dissipation and dispersion of electromagnetic energy by the medium. Finally, as discussed at the end of the previous section, the point where the frequencies change sign seems to be of critical importance, yet the physical origin and meaning of these changes in frequency seems unclear.

Physically, the notion of a negative frequency wave is a curious one. Naively, a frequency means how many times a process happens per second. The fact that this could occur a negative number of times seems to limit any immediate interpretation. Furthermore, if one "*zooms*" in on this medium, in the vacuum between its constituent particles, this change of sign cannot occur since here the speed of light is that of the vacuum, c. So it seems that these negative frequencies must be an artefact of this macroscopic refractive index n. Nevertheless, the point at which the frequencies change sign must, even at the microscopic level, present some interesting symptoms which hopefully may shed some light on the mechanisms at play. To this end, let us now consider the same two cases but using a microscopic model of a medium.

II.2 The Origin of Negative Frequencies

As intimated in the previous section, many of the effects that stem from the motion of media within the quantum vacuum occur in conjunction with the appearance of negative frequency modes of light. The analogy we considered bears strong resemblance, and in fact similar origins, to the well-known Cherenkov radiation where light is emitted in a cone consisting of waves with positive frequency in the laboratory frame, but negative frequency in the rest frame of the particle [17–20]. Both cases stem from the mixing of positive and negative frequency waves in the two materials [44, 74]. Furthermore, Hawking radiation [24] and its laboratory analogues [32, 33] originate from the change in the sign of the frequency of a wave as it crosses the event horizon, an effect that has been recently experimentally observed in a number of analogue systems [30, 75–77], and led to the identification of new pulse propagation phenomena within fibre optics [78].

Such situations are not trivial, and there appears to be a lack of any immediate physical interpretation for the appearance and significance of these negative frequency modes. In this section, heavily derived from a paper co-written by the author [1] , we will use a simple microscopic model of a dielectric medium—similar to that described in [79, 80]—to understand the microscopic meaning of the effects that arise when $v > c/\mathrm{n}$.

II.2.1 The Microscopic Model of a Homogeneous Dielectric

Our goal in this section is again to consider how the dynamics of a harmonic oscillator is modified as it travels through a medium. However, instead of assuming that the medium is homogeneous, we effectively “zoom” in and consider the medium from the perspective of its microscopic structure. Again, in the spirit of taking simplest cases first, we consider a one–dimensional model for a dielectric medium composed of equally spaced point–like scatterers (atoms) at positions $x = na$, each with a polarizability α (here α is independent of frequency). The

Lagrangian density for such a system is given by

$$\mathscr{L}_0 = -\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{\rho(x)}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 \tag{II.34}$$

where

$$\rho(x) = 1 + \alpha\sum_n \delta(x - na). \tag{II.35}$$

Applying the Euler-Lagrange equations (I.35) to (II.34) gives us the equations of motion for the field

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\rho(x)}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi(x,t) = 0 \tag{II.36}$$

and by expanding the scalar field into its frequency modes via

$$\varphi(x,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\varphi(x,\omega)e^{-\mathrm{i}\omega t} \tag{II.37}$$

and inserting into (II.36), we arrive at the Helmholtz equation (which is in essence the frequency space wave equation)

$$\left[\frac{d^2}{dx^2} + \frac{\omega^2\rho(x)}{c^2}\right]\varphi(x,\omega) = 0. \tag{II.38}$$

At each atom position, the wave is continuous, i.e.

$$\lim_{\eta\to 0}\left[\varphi(na-\eta) - \varphi(na+\eta)\right] = 0. \tag{II.39}$$

Meanwhile the boundary conditions on the derivative of φ can be found through integrating (II.38) over the infinitesimal interval $[na-\eta, na+\eta]$

$$\lim_{\eta\to 0}\left[\frac{\partial\varphi}{\partial x}\right]_{na-\eta}^{na+\eta} = -\frac{\omega^2}{c^2}\alpha\varphi(na). \tag{II.40}$$

Let's consider the unit cell centred around $x = 0$. For $x < 0$, we express each mode in terms of left and right going waves in the following form

$$\varphi(x) = \varphi_L\,\mathrm{e}^{\mathrm{i}k_0 x} + \varphi_R\,\mathrm{e}^{-\mathrm{i}k_0 x}, \tag{II.41}$$

with $\omega = ck_0$, and where the subscripts L/R stand for left or right travelling waves respectively. If we then apply the periodic boundary condition $\varphi(x+a) = \exp(iKa)\varphi(x)$ where K is the Bloch vector, we can write the mode when $x > 0$ as

$$\varphi(x) = \left(\varphi_L\, \mathrm{e}^{\mathrm{i}k_0(x+a)} + \varphi_R\, \mathrm{e}^{-\mathrm{i}k_0(x+a)}\right) e^{iKa}. \tag{II.42}$$

In order to find the form of the Bloch vector K, we use the boundary conditions (II.39–II.40) to find a pair of equations that must be satisfied by each mode

$$\begin{aligned} \varphi_L + \varphi_R &= \left(\varphi_L\, \mathrm{e}^{\mathrm{i}k_0 a} + \varphi_R\, \mathrm{e}^{-\mathrm{i}k_0 a}\right) e^{iKa} \\ (1+ik_0\alpha)\varphi_L - (1-ik_0\alpha)\varphi_R &= \left(\varphi_L\, \mathrm{e}^{\mathrm{i}k_0 a} - \varphi_R\, \mathrm{e}^{-\mathrm{i}k_0 a}\right) e^{iKa}. \end{aligned} \tag{II.43}$$

These two equations can be written as the matrix multiplication

$$\begin{pmatrix} 1 & 1 \\ k_0(i-k_0\alpha) & -k_0(i+k_0\alpha) \end{pmatrix} \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}k_0 a} & \mathrm{e}^{-\mathrm{i}k_0 a} \\ ik_0\mathrm{e}^{\mathrm{i}k_0 a} & -ik_0\mathrm{e}^{-\mathrm{i}k_0 a} \end{pmatrix} \begin{pmatrix} \varphi_L^1 \\ \varphi_R^1 \end{pmatrix} e^{iKa}. \tag{II.44}$$

from which we can establish the matrix equation

$$\mathbf{T} \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix} = e^{iKa} \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix} \tag{II.45}$$

where the transfer matrix $\mathbf{T}$ across a complete unit cell is given by

$$\mathbf{T} = \begin{bmatrix} \left(1 + i\frac{\alpha k_0}{2}\right) e^{-ik_0 a} & i\frac{\alpha k_0}{2} e^{-ik_0 a} \\ -i\frac{\alpha_0}{2} e^{ik_0 a} & \left(1 - i\frac{\alpha k_0}{2}\right) e^{ik_0 a} \end{bmatrix} \tag{II.46}$$

which leads to the eigenvalue equation

$$\left|\mathbf{T} - e^{iKa}\, \mathbb{1}_2\right| = 0 \tag{II.47}$$

where $\mathbb{1}_2$ is the 2×2 identity matrix. The solutions of this yield the dispersion relation that relates ω and the Bloch vector K

$$K = \pm\frac{1}{a}\arccos\left[\cos\left(k_{n,K}a\right) - \frac{\alpha k_{n,K}}{2}\sin\left(k_{n,K}a\right)\right]. \tag{II.48}$$

Note that the Bloch vector is in general a complex quantity, the real part of which is proportional to the phase shift between neighbouring unit cells. Figure II.3 shows this quantity plotted as a function of frequency for the case of $\alpha/a = 1$. For small $\omega a/c$ it takes the approximate form

$$K \sim \pm\sqrt{1 + \frac{\alpha}{a}}\,\frac{\omega}{c} = \pm\mathrm{n}\frac{\omega}{c} \tag{II.49}$$

which is the result that would be obtained for a macroscopic medium with the uniform permittivity $\epsilon = 1 + \alpha/a$, and with index $\mathrm{n} = \sqrt{\epsilon}$ (and is also the spatial average of the microscopic permittivity given in (II.38)). From now on, we will refer to this limit as the *macroscopic limit*. It is worth reiterating that here we are exploring the microscopic structure of the "homogeneous" refractive index case previously described to see what effects taking into account this microscopic structure has on the dynamics of the oscillator, and therefore how it affects our predictions of the spontaneous emission of this two level system as it moves through this medium.

The form of the wave modes derived above are summarised as

$$\varphi(x) = \begin{cases} \left(\varphi_L\,\mathrm{e}^{\mathrm{i}k_0(x-a/2)} + \varphi_R\,\mathrm{e}^{-\mathrm{i}k_0(x-a/2)}\right)e^{-iKa/2} & x < 0 \\ \left(\varphi_L\,\mathrm{e}^{\mathrm{i}k_0(x+a/2)} + \varphi_R\,\mathrm{e}^{-\mathrm{i}k_0(x+a/2)}\right)e^{iKa/2} & x > 0 \end{cases}$$

where we have arbitrarily made the change $\varphi_{L/R} \to \varphi_{L/R}\exp(-iKa/2)$ purely for convenience. To complete this determination of the form of the wave modes, consider again the the boundary conditions (II.39–II.40), from which we find

$$\varphi_R = \varphi_L\frac{\sin\left[(k_0 - K)a/2\right]}{\sin\left[(k_0 + K)a/2\right]}. \tag{II.50}$$

If we now set $\varphi_L = N_{n,K} \sin[(k_0 + K)a/2]$, we can write down form of the field $\varphi = \varphi_{n,K}$ in terms of the normalisation constant $N_{n,K}$ as

$$\varphi_{n,K}(x) = N_{n,K} \times \tag{II.51}$$

$$\begin{cases} \mathrm{e}^{-\mathrm{i}Ka/2}\left[\sin\left(a\frac{k_{n,K}+K}{2}\right)\mathrm{e}^{\mathrm{i}k_{n,K}(x+a/2)} + \sin\left(a\frac{k_{n,K}-K}{2}\right)\mathrm{e}^{-\mathrm{i}k_{n,K}(x+a/2)}\right] & x < 0 \\ \mathrm{e}^{\mathrm{i}Ka/2}\left[\sin\left(a\frac{k_{n,K}+K}{2}\right)\mathrm{e}^{\mathrm{i}k_{n,K}(x-a/2)} + \sin\left(a\frac{k_{n,K}-K}{2}\right)\mathrm{e}^{-\mathrm{i}k_{n,K}(x-a/2)}\right] & x > 0 \end{cases} \tag{II.52}$$

where the subscripts K and n label the Bloch vector and frequency band respectively. For a fixed value of K there are many possible frequencies $k_{n,K} = \omega/c$ that obey the dispersion relation of the lattice $\omega(K)$ and the subscript n indicates which of these we are considering.

To determine the normalization constant $N_{n,K}$ we write the field (II.52) in the following form

$$\varphi_{n,K}(x) = N_{n,K} u_{n,K}(x)\mathrm{e}^{\mathrm{i}Kx} \tag{II.53}$$

where $u_{n,K}$ is a periodic function of x with period a. The scalar field $\varphi_{n,K}$ and the periodic function $u_{n,K}$ are plotted in Figure II.4. The normalization is chosen such that

$$N_{n,K}N^{\star}_{m,K}\int_{-a/2}^{a/2}\rho(x)u_{n,K}(x)u^{\star}_{m,K}(x)dx = a\delta_{nm}. \tag{II.54}$$

After applying the dispersion relation (II.48) we find this to be

$$N_{n,K} = \sin^{-1/2}(k_{n,K}a)\left[\left(1+\frac{\alpha}{2a}\right)\sin(k_{n,K}a) + \frac{\alpha k_{n,K}}{2}\cos(k_{n,K}a)\right]^{-1/2}. \tag{II.55}$$

Note that with the normalization (II.54) we can construct a delta function as follows

$$\begin{aligned}\sum_n \int_{-\pi/a}^{\pi/a}\frac{dK}{2\pi}\,\rho(x')\varphi_{n,K}(x)\varphi^{\star}_{n,K}(x') &= \int_{-\pi/a}^{\pi/a}\frac{dK}{2\pi}\mathrm{e}^{\mathrm{i}K(x-x')}\sum_n \rho(x')u_{n,K}(x)u^{\star}_{n,K}(x') \\ &= \delta(x-x').\end{aligned} \tag{II.56}$$

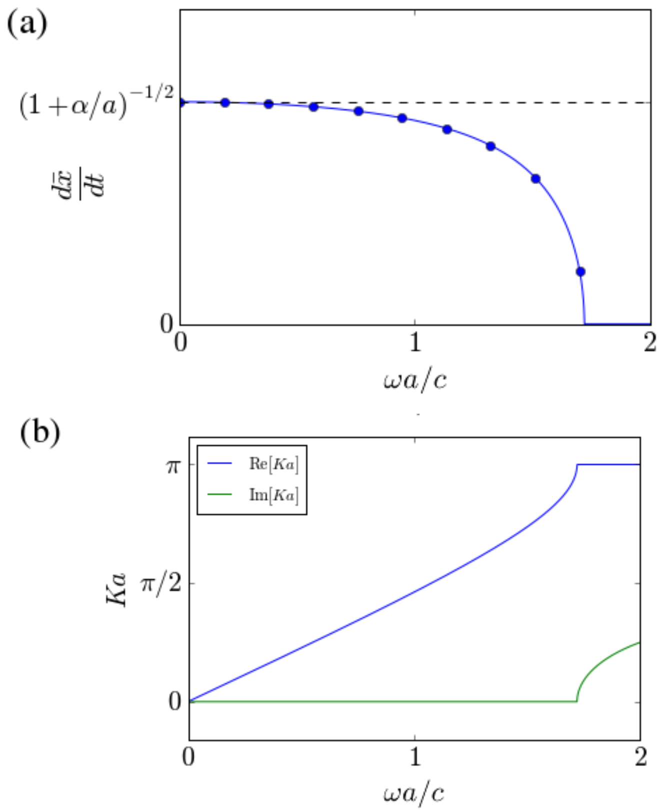

Figure II.3: The velocity of energy flow and the Bloch vector K in the lattice as a function of ω, plotted for $\alpha/a = 1$. Panel (a) shows the velocity of energy flow through the lattice (II.71) in units of c as a function of ω, for a frequency independent α. In the long wavelength limit the energy propagates at the phase velocity ω/k with the refractive index given by (II.49). As one approaches the Brillouin zone boundary ($K = \pi/a$) the energy flow velocity reduces to zero. The blue circles are the values of $\partial\omega/\partial K$ computed from (II.48), showing that the velocity of energy flow equals the group velocity. Panel (b) shows the real (upper) and imaginary (lower) parts of the Bloch wavevector as a function of ω, which is real for $K < \pi/a$.

To complete our description of our model medium we calculate the velocity of energy flow through the lattice. We begin with the expression for the time–averaged power flow through the medium $\langle S\rangle$. The units of φ are such that the power flow has the dimensions of energy per unit time. For a real monochromatic wave of frequency ω written in terms of its complex mode as

$$\varphi(x,t) = \varphi(x,\omega)e^{-\mathrm{i}\omega t} + \varphi^{\star}(x,\omega)e^{\mathrm{i}\omega t} \tag{II.57}$$

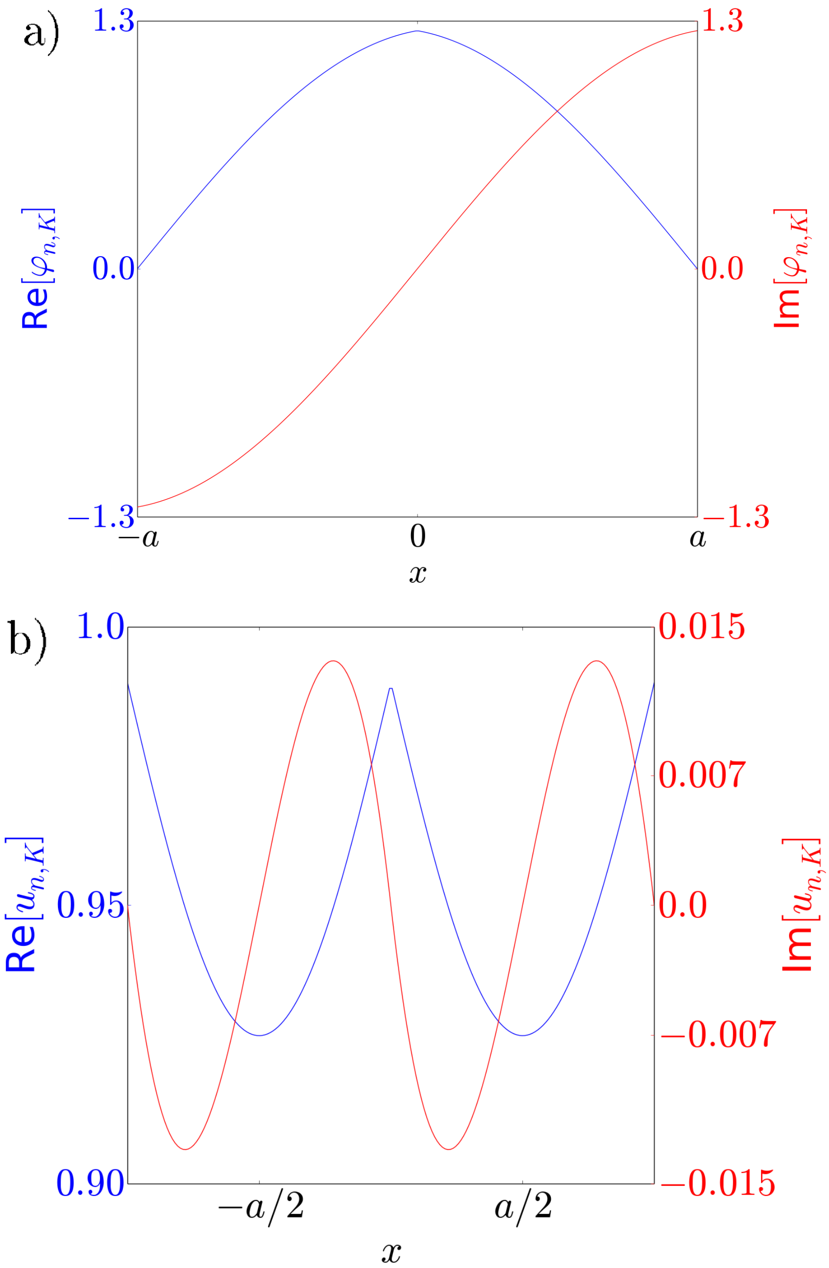

Figure II.4: a) Normalised scalar field $\varphi_{n,K}(x)$ plotted over two unit cells, where the Bloch vector is taken to be half the Brillouin zone width $K = \pi/2a$ with $n = 1$. b) Periodic function $u_{n,K}(x) = \varphi_{n,K}(x)\exp(-iKx)/N_{n,K}$ plotted over two unit cells with the same parameters as a).

the time averaged power flow is found via

$$\begin{aligned}\langle S\rangle &= -\langle \partial_t\varphi(x,t)\partial_x\varphi(x,t)\rangle = -\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt\,\partial_t\varphi(x,t)\partial_x\varphi(x,t)\\ &= \frac{\omega}{2}\mathrm{Im}\left[\varphi^\star(x,\omega)\frac{d\varphi(x,\omega)}{dx}\right]\end{aligned} \tag{II.58}$$

where we used the fact that the average of the time dependent parts integrates to zero

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{-T/2}^{T/2}dt\,e^{\pm\mathrm{i}2\omega t}=0 \tag{II.59}$$

and the time-independent parts average to a constant

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{-T/2}^{T/2}dt\;=1. \tag{II.60}$$

In the absence of dissipation, the power flow $\langle S\rangle$ must be independent of position or time indicating that the energy is neither lost or gained during the propagation. If we take the derivative of this flow, i.e

$$\frac{\partial\langle S\rangle}{\partial x}=\frac{\omega}{4i}\left[\varphi\left(x,\omega\right)\partial_x^2\varphi^{\star}\left(x,\omega\right)-\varphi^{\star}\left(x,\omega\right)\partial_x^2\varphi\left(x,\omega\right)\right] \tag{II.61}$$

and insert into it an expression for $\partial_x^2\varphi\left(x,\omega\right)$ derived from (II.38) and its complex conjugate, i.e.

$$\frac{d^2}{dx^2}\varphi^{\star}(x,\omega)=\frac{\omega^2\rho^{(\star)}(x)}{c^2}\varphi^{(\star)}(x,\omega) \tag{II.62}$$

we find that the gradient of the power flow is

$$\frac{\partial\langle S\rangle}{\partial x}=\frac{\omega^3}{2c^2}\varphi\left(x,\omega\right)\varphi^{\star}\left(x,\omega\right)\mathrm{Im}\left[\rho\left(x\right)\right]. \tag{II.63}$$

It is thus seen to be independent of x for real α. For the mode of the lattice given by (II.52) and normalized according to (II.54) the power flow is equal to

$$\langle S\rangle=\frac{1}{2a}\left(\frac{\omega}{c}\right)^2v_g \tag{II.64}$$

where the group velocity v_g is found via by taking the derivative of the frequency ω with respect to the Bloch vector K

$$\begin{aligned}v_g&=c\,\frac{dk_{n,K}}{dK}\\&=\frac{c\sin(k_{n,K}a)\sin(Ka)}{1-\cos(k_{n,K}a)\cos(Ka)+\frac{\alpha}{2a}\sin^2(k_{n,K}a)}.\end{aligned} \tag{II.65}$$

The power flow (II.64) is reduced relative to the power flow through free space (of a similarly normalized mode) due to the reduced group velocity $v_g < c$. As one would expect, the scatterers impede the propagation of power through the lattice. The velocity at which this energy flows through the lattice can also be calculated in this model. The power flow obeys a continuity equation,

$$\frac{\partial\langle S\rangle}{\partial x} + \frac{\partial\langle U\rangle}{\partial t} = 0 \tag{II.66}$$

where the time averaged energy density $\langle U\rangle$ is given by

$$\langle U\rangle = \frac{1}{2}\left\langle \left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{\rho(x)}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 \right\rangle \tag{II.67}$$

which in the limit of a monochromatic field becomes

$$\langle U\rangle \to \frac{1}{4}\left[\left|\frac{\partial\varphi}{\partial x}\right|^2 + \frac{\omega^2}{c^2}\rho(x)|\varphi|^2\right]. \tag{II.68}$$

Note that for a frequency dependent α the limiting form of $\langle U\rangle$ given in (II.68) would be given by the Brillouin energy density [17]. Using the continuity equation (II.66) we find that the centre of energy of the field,

$$\bar{x}(t) = \frac{\int dx x\langle U\rangle}{\int dx\langle U\rangle} \tag{II.69}$$

moves with a velocity given by the ratio of the total power to the total energy

$$\begin{aligned}\frac{d\bar{x}(t)}{dt} &= \frac{\int dx\, x\, \partial_t\langle U\rangle}{\int dx\, \langle U\rangle} - \frac{\int dx\, x\, \langle U\rangle \int dx\, \partial_t\langle U\rangle}{\left[\int dx\, \langle U\rangle\right]^2} \\ &= -\frac{\int dx\, x\, \partial_x\langle S\rangle}{\int dx\langle U\rangle} + \frac{\int dx\, x\, \langle U\rangle \int dx\, \partial_x\langle S\rangle}{\left[\int dx\, \langle U\rangle\right]^2} \\ &= \frac{-\left[x\langle S\rangle\right]_{-\infty}^{\infty} + \int dx\langle S\rangle}{\int dx\langle U\rangle} + \frac{\int dx\, x\, \langle U\rangle \left[\langle S\rangle\right]_{-\infty}^{\infty}}{\left[\int dx\, \langle U\rangle\right]^2} = \frac{\int dx\langle S\rangle}{\int dx\langle U\rangle}.\end{aligned} \tag{II.70}$$

assuming that the power flow at infinity vanishes. Using the monochromatic modes of the lattice (II.52) and expression (II.68) one obtains

$$\frac{d\bar{x}}{dt} = v_g \tag{II.71}$$

which reduces to c in the limit $\alpha \to 0$, and 0 in the limit $\alpha \to \infty$. The equality of the energy flow velocity with the group velocity can be found after an application of the dispersion relation (II.48).

The appearance of negative frequencies is ordinarily associated with a refractive index, a quantity which emerges in the macroscopic limit of this model medium, which is where $\omega a/c \ll 1$ and concerns the average properties of the wave. The spatial average of the wave over a unit cell centred on x is given by

$$\begin{aligned}\langle\varphi(x,t)\rangle &= \frac{\mathrm{e}^{-\mathrm{i}ck_{n,K}t}}{a}\int\limits_{-a/2}^{a/2}\varphi(x+y)\;dy\\ &= \frac{\mathrm{e}^{\mathrm{i}(Kx-ck_{n,K}t)}}{a}\int\limits_{-a/2}^{a/2}u_{n,K}(x+y)\mathrm{e}^{\mathrm{i}Ky}\\ &= \mathrm{e}^{\mathrm{i}(Kx-ck_{n,K}t)}\frac{1}{a}\sum_{m=0}^{\infty}\frac{(\mathrm{i}K)^m}{m!}\int\limits_{-a/2}^{a/2}y^m u_{n,K}(x+y)\;dy\\ &\sim \mathrm{e}^{\mathrm{i}(Kx-ck_{n,K}t)}\langle u_{n,K}(x)\rangle = A\,\mathrm{e}^{\mathrm{i}k_{n,K}(\mathrm{n}x-ct)}\end{aligned} \tag{II.72}$$

where A is a constant. Thus, for small $\omega a/c$ the average behaviour of the wave is as a plane wave with a wave-vector $\mathrm{n}\omega/c$ where n is given by (II.49). It is clear from figure II.3 that in this regime—with α independent of frequency—the medium behaves in a way that is equivalent to a dispersion-less homogeneous medium with index $\sqrt{1+\alpha/a}$.

That concludes the overview of our microscopic model. We may now proceed to consider the classical radiation damping and quantum emission rates analogously to the previous section.

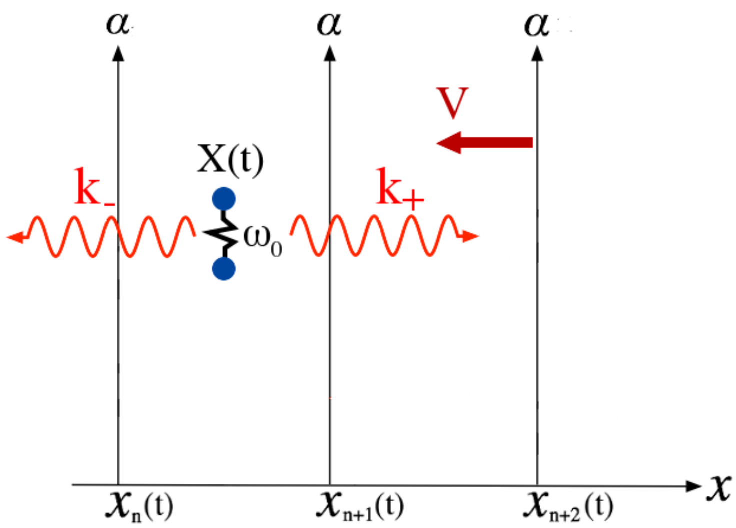

Figure II.5: In the frame where the lattice is in motion a small test particle is imagined to be embedded at a fixed position x_0. This particle has an internal degree of freedom $X(t)$ that behaves as a harmonic oscillator, coupled to the field φ. When the lattice is at rest the coupling to the field serves to damp the internal degree of freedom, as shown in panel b. Meanwhile when the lattice is moving, those frequencies that have changed sign between the rest frame and the moving frame serve to amplify the motion of the internal degree of freedom.

II.2.2 Classical Radiation Damping: A Microscopic Treatment

We will now consider the analogous case to that of section II.1.1 in which we established that when the oscillator moves through the medium faster than c/n, its radiation damping reduces to zero. The medium is now represented by a lattice of scattering particles, is it possible in this description to understand more deeply why this effect occurs? The oscillator stills obey the equation (II.7), but this time the scalar field obeys the wave equation

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{\rho(x')}{c^2}\frac{\partial^2}{\partial t'^2}\right]\varphi(x',t') = -\kappa\delta(x'-x_0')\dot{\mathsf{X}}(t'). \qquad \text{(II.73)}$$

The solutions to this equation are given by the sum of the solutions to the homogeneous equation (II.36) and the solutions generated by the RHS of (II.73) i.e

$$\varphi(x',t') = \varphi_H(x',t') - \kappa\int_{-\infty}^{\infty} dt_0{}'G(x_0',x_0',t',t_0')\dot{\mathsf{X}}(t_0'). \qquad \text{(II.74)}$$

where the Green's function G is a solution of the equation

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{\rho(x')}{c^2}\frac{\partial^2}{\partial t'^2}\right] G(x', x_0', t', t_0') = \delta(x' - x_0')\delta(t' - t_0'). \tag{II.75}$$

In this classical case, it is assumed that the scalar field is initially at rest and it is the initial amplitude of the oscillator that drives the oscillations of the system; we may therefore discard the homogeneous solutions to (II.74). Expression (II.74) is inserted into the oscillator equation of motion (II.7) and we find that in this case the equation of motion for the oscillator is considerably more complicated, being given by

$$\ddot{\mathsf{X}}(t') + \omega_0^2 \mathsf{X}(t') = \kappa^2 \frac{\partial}{\partial t'} \int_{-\infty}^{\infty} dt_0' G(x_0', x_0', t', t_0') \dot{\mathsf{X}}(t_0') \tag{II.76}$$

Let's first consider the frame where the lattice is at rest. To solve this equation we first look for the field of a source emitting at a fixed frequency ω, $G(x, x_0, \omega)$, which solves (II.75) for the case of a monochromatic field of frequency ω, and $\delta(x - x_0)$ on the right hand side. To find this quantity, we define an auxiliary function $G_K(x, x_0, \omega)$ via the integral expression

$$G(x, x_0, \omega) = \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi} G_K(x, x_0, \omega) \tag{II.77}$$

The quantity $G_K(x, x_0, \omega)$ is the wave produced from an array of equally spaced sources, each emitting at frequency ω and each source differing from the previous one by a phase $\exp(\mathrm{i}Ka)$,

$$\left[\frac{\partial^2}{\partial x^2} + k_0^2 \rho(x)\right] G_K(x, x_0, \omega) = a \sum_n \mathrm{e}^{\mathrm{i}Kna} \delta(x - x_0 - na). \tag{II.78}$$

The motivation for the ansatz (II.77) can be understood through considering the identity

$$\sum_n \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi} \exp(\mathrm{i}Kna) \delta(x - x_0 - na) = \frac{1}{a} \delta(x - x_0).$$

From (II.78) it is clear that $G_K(x+a, x_0, \omega) = \exp(\mathrm{i}Ka)G_K(x, x_0, \omega)$ and $G_K(x, x_0 + a, \omega) = \exp(-\mathrm{i}Ka)G_K(x, x_0, \omega)$ which implies that it can be written in the form

$$G_K(x, x_0, \omega) = \sum_{n,m} g_{n,m}(K, \omega)\mathrm{e}^{\mathrm{i}K(x-x_0)}\mathrm{e}^{\frac{2\pi\mathrm{i}}{a}(nx - mx_0)} \tag{II.79}$$

Substituting (II.79) in (II.78) we find

$$g_{n,m}(K, \omega) = \frac{\delta_{nm} - \frac{\alpha}{a}k_0^2 \sum_p g_{p,m}(K, \omega)}{k_0^2 - \left(K + \frac{2\pi n}{a}\right)^2}. \tag{II.80}$$

which is self–consistent if

$$\begin{aligned} \sum_n g_{n,m}(K, \omega) &= \left[1 + \frac{\alpha}{a}k_0^2 \sum_p \frac{1}{k_0^2 - \left(K + \frac{2\pi p}{a}\right)^2}\right]^{-1} \frac{1}{k_0^2 - \left(K + \frac{2\pi m}{a}\right)^2} \\ &= \frac{1}{D(\omega, K)\left[k_0^2 - \left(K + \frac{2\pi m}{a}\right)^2\right]} \end{aligned} \tag{II.81}$$

where $D(\omega, K)$ is defined in (II.84) below. Combining (II.77–II.81) then yields the Green's function at a fixed frequency

$$G(x, x_0, \omega) = \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi}\mathrm{e}^{\mathrm{i}K(x-x_0)} \left[\sum_n \phi_{n,K}(x - x_0) - \frac{\alpha}{a}\frac{k_0^2}{D(\omega, K)} \sum_{n,m} \phi_{n,K}(x)\phi_{m,K}(-x_0)\right] \tag{II.82}$$

where we defined

$$\phi_{n,K}(x) = \frac{\mathrm{e}^{\frac{2\pi\mathrm{i}n}{a}x}}{k_0^2 - \left(K + \frac{2\pi n}{a}\right)^2} \tag{II.83}$$

Evidently the full Green function (II.82) breaks up into two parts. After summation and integration the first term in the square brackets reduces to the free space Green function—it is the field of the source at x_0 in the absence of the lattice. The second term represents the response of the lattice to the source. It is clear from (II.82) that the response of the lattice is strongest when $D(\omega, K) = 0$, which is when the lattice dispersion relation (II.48) is fulfilled

$$D(\omega, K) = 1 + \frac{\alpha}{a}k_0^2 \sum_n \frac{1}{k_0^2 - \left(K + \frac{2\pi n}{a}\right)^2} = 1 - \frac{\alpha k_0 \sin(k_0 a)}{2\left[\cos(k_0 a) - \cos(Ka)\right]} \tag{II.84}$$

The time domain Green function $G(x, x_0, t, t_0)$ which we set out to obtain is the Fourier transform of (II.82)

$$G(x, x_0, t, t_0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(x, x_0, \omega) \mathrm{e}^{-\mathrm{i}\omega(t-t_0)} \tag{II.85}$$

To find the corresponding quantity in the frame where the lattice is in motion with velocity $-v$ we perform the Lorentz transformation (I.2) of both (x, t) and (x_0, t_0). This is

$$\begin{aligned} G(x', x'_0, t', t'_0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi} \mathrm{e}^{\mathrm{i}\gamma K(x'-x'_0+v(t'-t'_0))} \Big[\sum_n \phi_{n,K}(\gamma(x'-x'_0+v(t'-t'_0))) \\ - \frac{\alpha}{a} \frac{k_0^2}{D(\omega, K)} \sum_{n,m} \phi_{n,K}(\gamma(x'+Vt'))\phi_{m,K}(-\gamma(x'_0+vt'_0)) \Big] \mathrm{e}^{-\mathrm{i}\omega\gamma(t'-t'_0+v(x'-x'_0)/c^2)}. \end{aligned} \tag{II.86}$$

The particular case of $x' = x'_0$ reduces to

$$\begin{aligned} G(x'_0, x'_0, t', t'_0) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi} \Big[\sum_n \phi_{n,K}(\gamma v(t'-t'_0)) \\ - \frac{\alpha k_0^2}{aD(\Omega, K)} \sum_{n,m} \phi_{n,K}(\gamma(x'_0+vt'))\phi_{m,K}(-\gamma(x'_0+vt'_0)) \Big] \mathrm{e}^{-\mathrm{i}\gamma(\Omega-vK)(t'-t'_0)} \end{aligned} \tag{II.87}$$

with the function $\phi_{n,K}$ is defined by (II.83) and $D(\Omega, K)$ the dispersion relation for waves in the lattice.

The Green function (II.87) depends separately on t' and t'_0 rather than just $t' - t'_0$ as in the formulae preceding (II.14). This means that the damping of the oscillator is no longer simply related to the Fourier transform of (II.87). Nevertheless if we assume a fixed harmonic motion $\mathsf{X}(t'_0) = \cos(\omega t'_0)$, then the right hand side of (II.76) tells us the corresponding force $F(\omega, t')$ that would be required to maintain this motion. The work required to force an oscillator is usually found by integrating the force over a path taken by the oscillator which amounts to its extension (which is implicitly a function of time) $W = \int_{\mathsf{X}_0}^{\mathsf{X}_1} d\mathsf{X}\, F(\mathsf{X})$, this can also be expressed in terms of an integral over the time it takes to trace out the same path by a simple change of variables $W = \int_{t_0}^{t_1} dt\, F(t) d\mathsf{X}/dt$. The necessary rate of work required is then defined by $\dot{W}(\omega, t) = -\dot{\mathsf{X}}(t')F(\omega, t')$ which is found to be given by the integral

$$\dot{W}(\omega, t') = -\sin(\omega t')\omega^2\kappa^2 \frac{\partial}{\partial t'} \int_{-\infty}^{\infty} dt_0' G(x_0', x_0', t', t_0') \sin(\omega t_0'). \qquad \text{(II.88)}$$

If this requisite rate of work is averaged over a time interval $t' \in [-T/2, T/2]$ then one obtains a quantity equivalent to the damping of the oscillator as it moves through the lattice. For very long times compared to the dynamics of the lattice, i.e as $T \to \infty$, one obtains using (II.87),

$$\begin{aligned} \langle \dot{W}(\omega) \rangle &= -\frac{\omega^3\kappa^2}{2\gamma} \mathrm{Im}\left[\int_{-\infty}^{\infty} \frac{dK}{2\pi} \left(\frac{1}{\left(\frac{\omega_K}{c}\right)^2 - K^2} - \frac{\alpha}{a} \frac{(\omega_K/c)^2}{D(\omega_K, K)\left[\left(\frac{\omega_K}{c}\right)^2 - K^2\right]^2} \right) \right] \\ &= \frac{\alpha\omega^3\kappa^2}{4\gamma a} \sum_n \frac{s_n(\omega_{K_n}/c)^2}{\left(\frac{\partial D(\omega_K, K)}{\partial K}\right)_{K=K_n} \left[\left(\frac{\omega_{K_n}}{c}\right)^2 - K_n^2\right]^2} \end{aligned} \qquad \text{(II.89)}$$

where the second line was obtained by analytically continuing the integral over complex K similarly to that in Section II.1.1. The first term has no poles in the lower half plane and therefore integrates to zero. The second has a pole for every root of $D(\omega_K, K) = 0$ and thus an application of Cauchy's residue theorem gives the given result. We also used the definitions $\omega_K = \omega/\gamma + vK + \mathrm{i}\eta$ and $s_n = -\mathrm{sign}(\partial_\omega D(\omega_K, K)/\partial_K D(\omega_K, K))_{K=K_n}$, with K_n the roots of $D(\omega_K, K) = 0$. In figure II.6 the functions (II.88) and (II.89) are plotted to illustrate the correspondence between this lattice model of the medium, and the macroscopic description in terms of a refractive index. As is evident from panels II.6a and II.6b, the fact that the damping of the oscillator is predicted to reduce to zero (II.14) can be attributed to the collisions with the scatterers that make up the lattice. Each collision causes an emission of radiation which then acts back on the motion of the oscillation. *Above $v = c/n$ this radiation acts to both damp and amplify the oscillator in equal amounts*.

In panels II.6c and II.6d the time averaged rate of work is plotted for two different frequencies of oscillation and shows that for low frequencies the average work required decreases rapidly to zero for velocities $v > c/\mathrm{n}$ as predicted in the limit where the medium can be described with a refractive index (II.49), as shown in (II.14). For the higher frequencies (figure II.6d) the macroscopic limit is less applicable and the oscillator remains significantly damped for velocities above $v = c/\mathrm{n}$. In this regime notice that there is still a sharp reduction in the rate of work required, but that it occurs at a lower velocity. This happens when, for one of the roots K_n, $(\partial D(\omega_K, K)/\partial K)_{K=K_n}$ passes through zero, which in general happens when the velocity of the lattice equals the *group velocity* of the waves. Therefore as the structure of the lattice becomes more apparent to the waves, it is not the phase, but the group velocity which is important for the radiation damping.

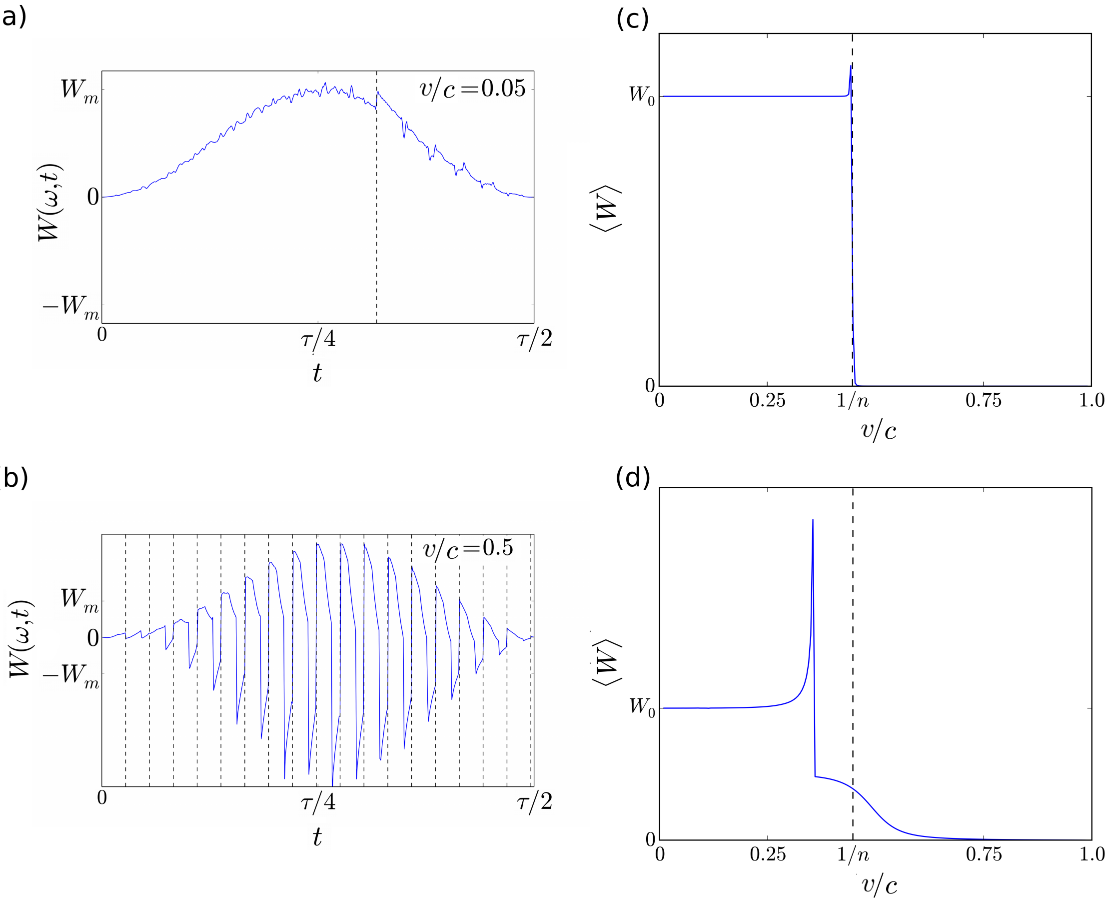

Figure II.6: Rate of work done to move an oscillator with a constant amplitude and frequency through a lattice of scatterers ($\alpha = 4.0$). Panels (a–b) show the work required as a function of time computed from (II.88) for half a period of oscillation $\tau = \pi/\omega = \pi a/c$. The maximum value of the work required to maintain the motion of the oscillator when $v/c = 0$ is given by W_m. As the velocity of the oscillator is increased from zero, sharp features appear in the work as a function of time. These are due to the collisions with the scatterers, occuring at the times shown by the vertical dashed lines. The collisions result in the emission of sharp wavefronts that reflect off the lattice and act back on the oscillator, resulting in the sequence of sharp features in panel (a). As the velocity is increased beyond $v = c/\mathrm{n} = c/\sqrt{1 + \alpha/a} = c/\sqrt{5}$ the collisions with the scatterers result in a field that acts back on the oscillator, adding and subtracting energy in equal amounts. Panel (b) shows that in this regime the requisite work oscillates, averaging to zero as predicted in the macroscopic limit. Panels (c–d) show two plots of the time averaged work (II.89) as a function of velocity, with W_0 being the average work required when the oscillator is at rest (in the macroscopic limit $W_0 = \Gamma\omega^2$). In panel (c) the frequency is $\omega = 0.0005c/a$ and the work decreases in a step–like fashion as the velocity is increased above c/n (as in the macroscopic limit shown in figure II.1), meanwhile panel (d) shows a higher oscillation frequency $\omega = 0.1c/a$ where the macroscopic limit is less applicable, and the work less rapidly reduces to zero.

II.2.3 Quantum Mechanical Absorption Rate in a Lattice

In section II.1.2, we considered the quantum mechanics of a dipole travelling through a homogeneous medium of refractive index n and found that when the relative motion between the dipole and the medium exceeds the speed of light within the medium, the dipole is predicted to jump into its first excited state. Here, we consider the analogous case where the medium is no longer homogeneous but is instead modelled by the discrete array of point like scatterers of the previous section. Similarly to the previous section, we find that the negative frequency modes present in the homogeneous case are missing and that the dynamics are somewhat more complicated.

Using the same coupling term between the field and the dipole as section (II.1.2), the only difference is that we now employ the microscopic scalar field Lagrangian density (II.34). The Hamiltonian of the system can then be constructed from the canonical variables derived from the Lagrangian. The canonical momentum of the field (defined in terms of the Lagrangian density $\mathscr{L}$) this time is given by

$$\Pi_{\varphi} = \frac{\partial \mathscr{L}}{\partial(\partial_t \varphi)} = \frac{\rho(x)}{c^2}\frac{\partial \varphi}{\partial t} - \kappa \mathsf{X} \delta(x - vt) \tag{II.90}$$

and that of the oscillator is again given by (II.21). The Hamiltonian for a dipole moving through an array of point-like scatterers with velocity v can be written in terms of these canonical variables using (I.37)

$$\begin{aligned} H &= \int_{-\infty}^{\infty} dx \left(\Pi_{\varphi} \cdot \frac{\partial \varphi}{\partial t}\right) + p_{\mathsf{X}} \cdot \frac{\partial \mathsf{X}}{\partial t} - L \\ &= \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{c^2}{2\rho(x)} \left[\Pi_{\varphi} + \kappa \mathsf{X} \delta\left(x - vt\right)\right]^2 + \kappa \mathsf{X} \delta\left(x - vt\right) v \frac{\partial \varphi}{\partial x} \right\} \\ &+ \frac{1}{2\gamma}\left(p_{\mathsf{X}}^2 + \omega_0^2 \mathsf{X}^2\right) \end{aligned} \tag{II.91}$$

Although it does not effect the validity of our results, it is slightly difficult to make sense of this Hamiltonian because we have to—for instance—divide by $\rho(x)$ which contains delta functions. Where this is problematic we must understand the delta functions as the limit of a sharply peaked function.

The system is quantised by rewriting the canonical variables (φ, Π_φ) as operators $(\hat{\varphi}, \hat{\Pi}_\varphi)$ in the Hamiltonian (II.91) and enforcing the commutation relations (I.38) and (I.29). The Hamiltonian operator (II.91) can be split into a non–interacting part $\hat{H}_0$ and an interaction term $\hat{H}_I$ as in Section I.5 where to leading order in κ the interacting part is given as

$$\hat{H}_I = \kappa \int_{-\infty}^{\infty} dx\, \hat{\mathsf{X}}\, \delta\left(x - vt\right) \left[\frac{c^2}{\rho(x)}\hat{\Pi}_\varphi + v\frac{\partial \hat{\varphi}}{\partial x}\right]. \tag{II.92}$$

Note that this is identical to (II.23) except for the replacement of the refractive index n by $\sqrt{\rho(x)}$. We work in the interaction picture (see Section I.5 for details), and the field operator $\hat{\varphi}$ and its conjugate variable $\hat{\Pi}_\varphi$ are written as

$$\hat{\varphi}(x,t) = \sum_n \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi}\sqrt{\frac{\hbar c^2}{2\omega_{n,K}}}\left[u_{n,K}\left(x\right) e^{i\left(Kx - \omega_{n,K}t\right)}\hat{a}_{n,K} + \text{h.c}\right] \tag{II.93}$$

and

$$\hat{\Pi}_\varphi(x,t) = -\frac{i\rho(x)}{c^2}\sum_n \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi}\sqrt{\frac{\hbar c^2 \omega_{n,K}}{2}}\left[u_{n,K}\left(x\right) e^{i\left(Kx - \omega_{n,K}t\right)}\hat{a}_{n,K} - \text{h.c}\right] \tag{II.94}$$

where $\omega_{n,K}$ are the solutions to the dispersion relation of waves in the lattice (II.48) and the functions $u_{n,K}$ are normalized according to (II.54). Meanwhile the oscillator operators $\hat{\mathsf{X}}$ and $\hat{p}_{\mathsf{X}}$ are given by (I.31) where the frequencies are now the contracted frequencies $\omega_0 \to \omega_0/\lambda$ due to the time dilation, where λ is the Lorentz factor (I.4). The creation and annihilation operators for the fields obey the relations

$$\left[\hat{a}_{n,K}, \hat{a}^\dagger_{m,K'}\right] = 2\pi\delta(K - K')\delta_{nm}. \tag{II.95}$$

We expand the wave function for the system to first order in the interaction Hamiltonian (see Section I.5), considering transitions of the system away from the ground state

$$|\psi(t)\rangle = |0\rangle_{\mathsf{X}} \otimes |0\rangle_{\varphi} + \sum_n \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi}\, \xi_{n,K}(t)\, |1\rangle_{\mathsf{X}} \otimes |1_{n,K}\rangle_{\varphi} \tag{II.96}$$

where $\xi_{n,K}$ is the time dependent expansion coefficient which is to be determined. In order to evaluate the derivative of the scalar field amplitude φ with respective to position appearing in expression (II.92), we write the periodic function $u_{n,K}(x)$ as a sum over the diffracted orders via

$$u_{n,K}(x) = \sum_m u_{n,K}^m \mathrm{e}^{i\frac{2\pi m}{a}x} \tag{II.97}$$

where the coefficients $u_{n,K}^m$ are found by evaluating the integral

$$u_{n,K}^m = \frac{1}{a} \int_{-a/2}^{a/2} dx\ u_{n,K}(x)\, e^{-i\frac{2\pi m}{a}x}. \tag{II.98}$$

Proceeding analogously to Section II.1.2, the rate of change of the quantity $\xi_{n,K}$ is found to be given by

$$\dot{\xi}_{n,K}(t) = \frac{i\kappa c}{2} \sum_m \frac{\omega_{n,K} - v\left(K - \frac{2\pi m}{a}\right)}{\sqrt{\omega_0 \omega_{n,K}}} u_{n,K}^{m\star} e^{-i\left[\left(K - \frac{2\pi m}{a}\right)v - \omega_{n,K} - \frac{\omega_0}{\gamma}\right]t} \tag{II.99}$$

and $\xi_{n,K}$, at the end of the interaction period running from $-T/2$ to $T/2$, is therefore

$$\xi_{n,K}(T/2) = \frac{\kappa c}{2\sqrt{\omega_0 \omega_{n,K}}} \sum_m u_{n,K}^{m\star} \left[\omega_{n,K} - v\left(K - \frac{2\pi m}{a}\right)\right] \times \tag{II.100}$$

$$\frac{\sin\left[\left(\left(K - \frac{2\pi m}{a}\right)v - \omega_{n,K} - \frac{\omega_0}{\gamma}\right)T/2\right]}{\left[\left(K - \frac{2\pi m}{a}\right)v - \omega_{n,K} - \frac{\omega_0}{\gamma}\right]} \tag{II.101}$$

Again, we take the absolute value of this quantity and divide by the total time T, yielding

$$\frac{\left|\int_{-T/2}^{T/2} dt\,\dot{\xi}_{n,K}(t)\right|^2}{T} = \frac{\kappa^2 c^2}{4\omega_0\omega_{n,K}T}\times$$
$$\sum_{m,m'}\left[\omega_{n,K} - v\left(K - \frac{2\pi m}{a}\right)\right]\left[\omega_{n,K} - v\left(K - \frac{2\pi m'}{a}\right)\right]^{\star} u^m_{n,K}\, u^{m'\star}_{n,K}\times$$
$$\frac{\sin\left[\left(\left(K - \frac{2\pi m}{a}\right)v - \omega_{n,K} - \frac{\omega_0}{\gamma}\right)T/2\right]\sin\left[\left(\left(K - \frac{2\pi m'}{a}\right)v - \omega_{n,K} - \frac{\omega_0}{\gamma}\right)T/2\right]}{\left[\left(K - \frac{2\pi m}{a}\right)v - \omega_{n,K} - \frac{\omega_0}{\gamma}\right]\left[\left(K - \frac{2\pi m'}{a}\right)v - \omega_{n,K} - \frac{\omega_0}{\gamma}\right]} \tag{II.102}$$

In the limit of long times, as $T \to \infty$, only the terms where $m = m'$ will survive and we therefore immediately write the full emission rate as

$$\begin{aligned}\Gamma_{0\to1} = \frac{\kappa^2 c^2}{4}\sum_{n,m}\int_{-\pi/a}^{\pi/a}\frac{dK}{2\pi}\frac{\left[\omega_{n,K} - v\left(K - \frac{2\pi m}{a}\right)\right]^2}{\omega_0\omega_{n,K}}\left|u^m_{n,K}\right|^2 \\ \times\,\delta\left[\left(K - \frac{2\pi m}{a}\right)v - \omega_{n,K} - \frac{\omega_0}{\gamma}\right]\end{aligned} \tag{II.103}$$

We use the fact that the coefficients obey $u^m_{n,K} = u^0_{n,K+2\pi m/a}$ to extend the integral over the Bloch vector K

$$\Gamma_{0\to1} = \frac{\kappa^2 c^2}{4\omega_0}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dK}{2\pi}\frac{\left(\omega_{n,K} - vK\right)^2}{\omega_{n,K}}\left|u_{n,0}(K)\right|^2\delta\left(\omega_{n,K} - Kv + \frac{\omega_0}{\gamma}\right) \tag{II.104}$$

Thus, after carrying out the integral over the Bloch vector, the rate at which the oscillator extracts energy from the lattice is given by

$$\Gamma_{0\to1} = \frac{\kappa^2 c^2\omega_0}{4\gamma^2}\sum_n\frac{1}{\omega(K_n)}\frac{\left|u_0(K_n)\right|^2}{\left|v - \partial\omega(K)/\partial K\right|_{K=K_n}} \tag{II.105}$$

Although these ‘negative frequencies’ are somewhat mysterious in the macroscopic limit, in this model system they are simply a shorthand for the periodic interaction of the oscillator with the lattice of scatterers as they move past. In the limit of small a, only the first frequency band ($n = 1$) contributes significantly to (II.105), and a good approximation to the solution to the dispersion relation is $K = \mathrm{n}k_{1,K}$ where the index is that given in (II.49). In this regime the rate of energy absorption reduces to

$$\Gamma_{0\to 1} \sim \frac{c\kappa^2}{4\mathrm{n}\gamma}\Theta\left(\frac{v\mathrm{n}}{c}-1\right) \tag{II.106}$$

which is the negative part of the damping constant that we found in the classical treatment of this problem (II.14) (the factor of γ is present because we have worked in the frame where the oscillator is in motion). Thus the modes that have changed the sign of their frequency between reference frames (positive in the rest frame of the medium, and negative in the oscillator's rest frame) can serve to excite a relatively moving quantum system above its ground state, even though the field is initially in its ground state. The second mode that contributes positively to the damping in the classical result (II.14) does not contribute here because a quantum system cannot be reduced in energy below its ground state. Having derived the result (II.106) from a model of a dielectric medium as a lattice of scatterers we can—as in the previous section—now understand these excitations of the oscillator as being due to the modified radiation damping force that the oscillator is subject to as it moves through the lattice (see Figure II.7). When the velocity of the oscillator is above c/n then the periodic interaction with the scatterers is such that the radiation reaction can equally both amplify and damp the vibration of the oscillator above the ground state (rather than just damp the motion as it does when $v = 0$). In this quantum mechanical case, the amplifying part of this force then serves to excite the oscillator above the ground state.

It should be noted that in this example, the relative velocity between the medium and the oscillator is held constant throughout the excitation. In a similar model [44], it is thought that taking into account the changes in the kinetic energy of the centre of mass would help show the role of these negative modes. The change in the sign of the frequency, and therefore the change in the sign of the energy packet absorbed by the medium is equivalent to subtracting a positive energy packet, i.e decreasing the overall relative kinetic energy of the two entities. In this sense, this model is somewhat analogous to quantum friction [8–12] and the negative modes that appear in such calculations can be taken to be from the same origin as those explored here.

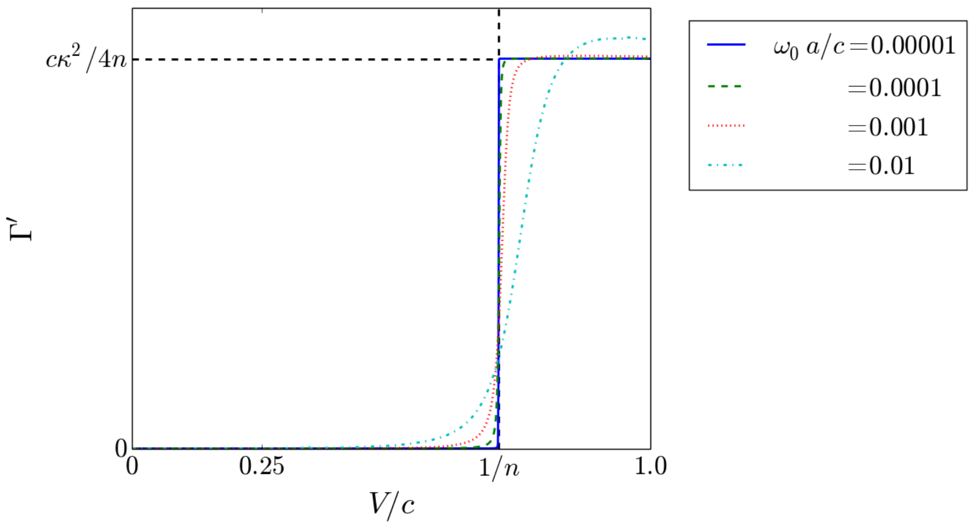

Figure II.7: The absorption rate (II.105) plotted for $\alpha/a = 1$ for various oscillator frequencies taken in the frame of the oscillator, $\Gamma' = \gamma\Gamma_{0\to1}$. For $\omega_0\, a/c = 1.0\times10^{-5}$, the absorption rate exhibits the step-like increase with velocity given in equation (II.106), representing the macroscopic limit of a homogeneous material of refractive index $n = \sqrt{1+\alpha/a}$. As $\omega_0\, a/c$ is increased, the effect of the lattice becomes more prominent and the velocity above which there is a non-zero absorption decreases. The effect of including the lattice is therefore to smooth of this step-like rate and allow absorption of energy for velocities much lower than the phase velocity of light in the medium originally suggested by the macroscopic limit.

II.3 The Vacuum Energy Outside a Moving Dielectric

All of the cases considered so far involve particles travelling through media, but in reality most systems of interest will involve separate, non-overlapping bodies. The emission or absorption of a photon by the two-level systems considered up until now have relied on a coupling to the vacuum field. But if these two-levels systems had been *outside* the moving bodies, would there still have been absorption or emission of a photon? For this to be possible, it seems that the vacuum field outside the medium, at the position of the atom, must be modified somehow by the motion of the dielectric.

In this section, we investigate this notion by considering the form of the vacuum energy in the vicinity of a moving infinite half plane dielectric contained in the region $z < 0$. The space where $z > 0$ is vacuum. Although we now consider the full electromagnetic vector fields in 3 dimensions; for simplicity, we continue to use non-dispersive, absorption-less media, that is to say the refractive index n > 1 is taken to be a *real* constant (we consider the approximation in which it is independent of the frequency.) In the next chapter, we will consider the physics of moving dielectrics in the presence of absorbing, dispersive media.

Firstly, we consider the quantisation of the stationary case. It is thought that along the surface of moving dielectrics, there will be a finite flow of energy along the surface of the dielectric that decays away from the moving body (see for example [44]). The quantity of interest here will therefore be the Poynting vector outside the half space dielectric.

Since this decaying energy will take the form of evanescent waves we turn to the model developed by Carniglia and Mandel [81] which allows the quantisation of evanescent waves at a planar dielectric boundary. Once we have shown that, as one would expect, the Poynting vector is zero in the stationary case, we will then derive the corresponding moving case.

II.3.1 Stationary Media

The model developed by Carniglia and Mandel [81] treats the quantisation of the electromagnetic field at a planar boundary at $z = 0$ between free space $(z > 0)$ and a dielectric. It allows for the possibility that, for large angles of incidence from within the medium, the modes undergo total internal reflection resulting in the appearance of evanescent waves along the moving surface. The completeness of these modes is shown in a separate paper [82]. The boundary conditions set the relationship between the modes on either side of the surface. For a given vacuum wave vector $\mathbf{K}$, the equivalent wave vector in the medium is $\mathbf{k}$ where

$$\begin{aligned} K_x &= k_x \\ K_y &= k_y \\ K &= k/\mathrm{n} \end{aligned} \tag{II.107}$$

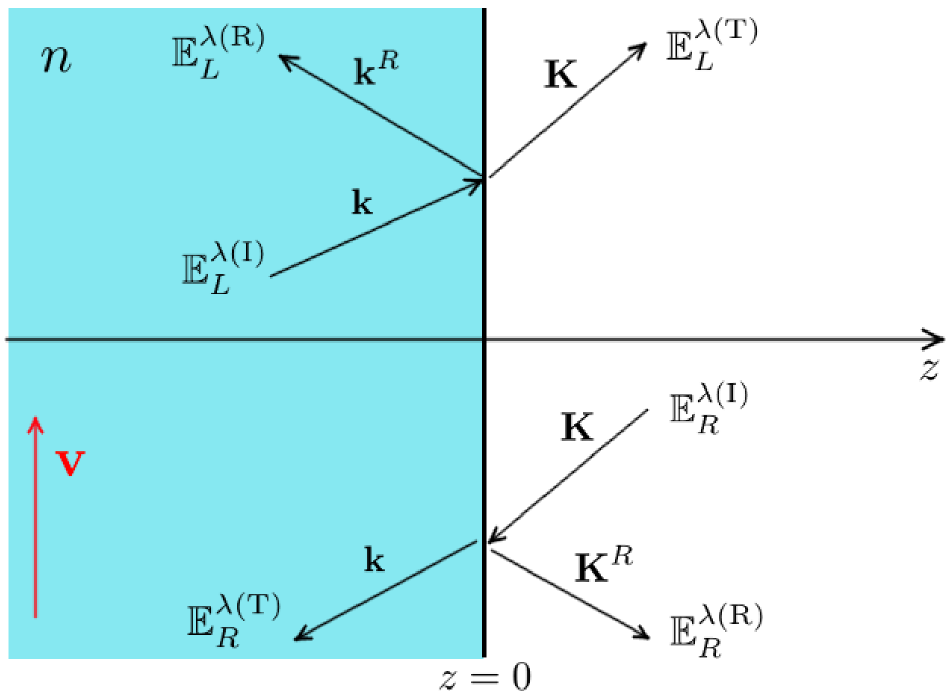

Figure II.8: Half space dielectric with refractive index $\mathrm{n}(\mathbf{x}, \omega) = n$ for $z \leq 0$ and vacuum for $z > 0$.

where $K = \omega/c$ is the frequency of the mode and the z-components are given in terms of the other components by

$$\begin{aligned} K_z &= \pm\sqrt{K^2 - K_x^2 - K_y^2} \\ k_z &= \pm\sqrt{k^2 - k_x^2 - k_y^2}. \end{aligned} \tag{II.108}$$

We use the same convention described in [81] that the sign of K_z is picked to match the sign of k_z. The electric field operator is expanded similarly to the vacuum expansion (I.40) except that now we must be careful to differentiate between modes whose origin is either to the left or the right of the boundary, there denoted $\mathbb{E}_\mathrm{L}^\lambda$ and $\mathbb{E}_\mathrm{R}^\lambda$ respectively. Again, the superscript λ denotes the polarisation (either transverse electric or transverse magnetic). Mathematically, this is expressed as

$$\hat{\mathbf{E}}(\mathbf{x},t) = \sum_{\lambda=1}^{2} \int\limits_{k_z>0} \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{cK}\left[\mathbb{E}_{\mathsf{L}}^{\lambda}(\mathbf{k},\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega(K)t}\,\hat{u}_\lambda(\mathbf{k}) + \mathsf{h.c}\right] \\ + \sum_{\lambda=1}^{2} \int\limits_{K_z<0} \frac{d^3\mathbf{K}}{(2\pi)^3}\sqrt{cK}\left[\mathbb{E}_{\mathsf{R}}^{\lambda}(\mathbf{K},\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega(K)t}\,\hat{v}_\lambda(\mathbf{K}) + \mathsf{h.c}\right] \tag{II.109}$$

where instead of (I.42), we now have a set of bosonic creation and annihilation operators for the modes from the left $\hat{u}_\lambda$ and the right $\hat{v}_\lambda$ that respectively satisfy the commutation relations

$$\begin{aligned} \left[\hat{u}_{\lambda_1}(\mathbf{k_1}), \hat{u}^\dagger_{\lambda_2}(\mathbf{k_2})\right] &= (2\pi)^3\,\hbar\delta_{\lambda_1\lambda_2}\delta^{(3)}\left(\mathbf{k_1}-\mathbf{k_2}\right) \\ \left[\hat{v}_{\lambda_1}(\mathbf{K_1}), \hat{v}^\dagger_{\lambda_2}(\mathbf{K_2})\right] &= (2\pi)^3\,\hbar\delta_{\lambda_1\lambda_2}\delta^{(3)}\left(\mathbf{K_1}-\mathbf{K_2}\right) \\ \left[\hat{u}_{\lambda_1}(\mathbf{K_1}), \hat{v}_{\lambda_2}(\mathbf{K_2})\right] &= 0. \end{aligned} \tag{II.110}$$

For the magnetic field, we have the analogous expression

$$\hat{\mathbf{B}}(\mathbf{x},t) = \sum_{\lambda=1}^{2} \int\limits_{k_z>0} \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{\frac{K}{c}}\left[\mathbb{B}_{\mathsf{L}}^{\lambda}(\mathbf{k},\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega(K)t}\,\hat{u}_\lambda(\mathbf{k}) + \mathsf{h.c}\right] \\ + \sum_{\lambda=1}^{2} \int\limits_{K_z<0} \frac{d^3\mathbf{K}}{(2\pi)^3}\sqrt{\frac{K}{c}}\left[\mathbb{B}_{\mathsf{R}}^{\lambda}(\mathbf{K},\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega(K)t}\,\hat{v}_\lambda(\mathbf{K}) + \mathsf{h.c}\right]. \tag{II.111}$$

The expansion coefficients take a different form in the two regions $z < 0$ and $z > 0$ and for waves that are incident from $z = -\infty$ and $z = +\infty$. In this work, we only wish to analyse the field in the vacuum outside of the dielectric. In this region, the expansion coefficients originating within the medium, at $z = -\infty$ are given in [81] as

$$
\begin{aligned}
\mathbb{E}_L^1(\mathbf{k},\mathbf{x}) &= \frac{1}{\sqrt{2}}\hat{\mathbf{e}}(\mathbf{k})\; t_L^1(k_z,K_z)\;\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{x}} \\
\mathbb{E}_L^2(\mathbf{k},\mathbf{x}) &= -\frac{1}{\sqrt{2}}\left[\hat{\mathbf{K}}\times\hat{\mathbf{e}}(\mathbf{k})\right]\; t_L^2(k_z,K_z)\;\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{x}} \\
\mathbb{B}_L^1(\mathbf{k},\mathbf{x}) &= \frac{1}{\sqrt{2}}\left[\hat{\mathbf{K}}\times\hat{\mathbf{e}}(\mathbf{k})\right] t_L^1(k_z,K_z)\,\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{x}} \\
\mathbb{B}_L^2(\mathbf{k},\mathbf{x}) &= \frac{1}{\sqrt{2}}\hat{\mathbf{e}}(\mathbf{k})\, t_L^2(k_z,K_z)\,\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{x}}
\end{aligned}
\tag{II.112}
$$

and those originating in the vacuum, from $z=\infty$ are listed as

$$
\begin{aligned}
\mathbb{E}_R^1(\mathbf{K},\mathbf{x}) &= \frac{1}{\sqrt{2}}\hat{\mathbf{e}}(\mathbf{k})\left[\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{x}} + r_R^1(k_z,K_z)e^{i\mathbf{K}^R\cdot\mathbf{x}}\right] \\
\mathbb{E}_R^2(\mathbf{K},\mathbf{x}) &= -\frac{1}{\sqrt{2}}\left\{\left[\hat{\mathbf{K}}\times\hat{\mathbf{e}}(\mathbf{k})\right]\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{x}} + \left[\hat{\mathbf{K}}^R\times\hat{\mathbf{e}}(\mathbf{k})\right] r_R^2(k_z,K_z)\mathrm{e}^{\mathrm{i}\mathbf{K}^R\cdot\mathbf{x}}\right\} \\
\mathbb{B}_R^1(\mathbf{K},\mathbf{x}) &= \frac{1}{\sqrt{2}}\left\{\left[\hat{\mathbf{K}}\times\hat{\mathbf{e}}(\mathbf{k})\right]\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{x}} + \left[\hat{\mathbf{K}}^R\times\hat{\mathbf{e}}(\mathbf{k})\right] r_R^1(k_z,K_z)\mathrm{e}^{\mathrm{i}\mathbf{K}^R\cdot\mathbf{x}}\right\} \\
\mathbb{B}_R^2(\mathbf{K},\mathbf{x}) &= \frac{1}{\sqrt{2}}\hat{\mathbf{e}}(\mathbf{k})\left[\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{x}} + r_R^2(k_z,K_z)\mathrm{e}^{\mathrm{i}\mathbf{K}^R\cdot\mathbf{x}}\right]
\end{aligned}
\tag{II.113}
$$

where the notation $\mathbf{K}^R$ indicates a reversal of the sign of K_z. The polarisation vectors are defined via

$$
\begin{aligned}
\mathbf{e}\left(\mathbf{k}\right) &= \hat{\mathbf{z}}\times\hat{\mathbf{K}}_{||} = -\frac{K_y}{K_{||}}\hat{\mathbf{x}} + \frac{K_x}{K_{||}}\hat{\mathbf{y}} \\
\hat{\mathbf{K}}\times\mathbf{e}\left(\mathbf{k}\right) &= -\frac{K_xK_z}{KK_{||}}\hat{\mathbf{x}} - \frac{K_yK_z}{KK_{||}}\hat{\mathbf{y}} + \frac{K_{||}}{K}\hat{\mathbf{z}} \\
\hat{\mathbf{K}}^R\times\mathbf{e}\left(\mathbf{k}\right) &= \frac{K_xK_z}{KK_{||}}\hat{\mathbf{x}} + \frac{K_yK_z}{KK_{||}}\hat{\mathbf{y}} + \frac{K_{||}}{K}\hat{\mathbf{z}}
\end{aligned}
\tag{II.114}
$$

where $K_{||}^2 = K_x^2 + K_y^2$. We also define the unit vector in the direction of $\mathbf{K}$ by $\hat{\mathbf{K}} = \frac{K_x}{K}\hat{\mathbf{x}} + \frac{K_y}{K}\hat{\mathbf{y}} + \frac{K_z}{K}\hat{\mathbf{z}}$. The reflection and transmission coefficients are the Fresnel coefficients for $\mu = 1$ [83]

$$
\begin{aligned}
t_L^1(k_z,K_z) &= \frac{1}{\mathsf{n}}\frac{2k_z}{k_z+K_z} \\
t_L^2(k_z,K_z) &= \frac{2k_z}{k_z+\mathsf{n}^2K_z} \\
r_R^1(k_z,K_z) &= \frac{K_z-k_z}{K_z+k_z} \\
r_R^2(k_z,K_z) &= \frac{\mathsf{n}^2K_z-k_z}{\mathsf{n}^2K_z+k_z}.
\end{aligned}
\tag{II.115}
$$

Substituting in the expressions for the electric and magnetic field operators (II.109) and (II.111) and applying the commutation relations (II.110), allows us to write the energy density as a sum over two kinds of modes

$$
\begin{aligned}
\langle\hat{\rho}\rangle = \frac{\epsilon_0\hbar c}{2}\int\limits_{k_z>0}\frac{d^3\mathbf{k}}{(2\pi)^3}\,K\,\left[\left|t_L^1\left(k_z,K_z\right)\ e^{i\mathbf{K}\cdot\mathbf{x}}\right|^2+\left|t_L^2\left(k_z,K_z\right)\ e^{i\mathbf{K}\cdot\mathbf{x}}\right|^2\right] \\
+\frac{\epsilon_0 c}{2}\int\limits_{K_z<0}\frac{d^3\mathbf{K}}{(2\pi)^3}\,K\,\Bigg[2+\left|r_R^1\left(k_z,K_z\right)\right|^2+\left|r_R^2\left(k_z,K_z\right)\right|^2 \\
+\frac{K_{\parallel}^2}{K^2}\mathsf{Re}\left[r_R^1\left(k_z,K_z\right)e^{-iK_zz}+r_R^2\left(k_z,K_z\right)e^{-iK_zz}\right]\Bigg]
\end{aligned}
\tag{II.116}
$$

When considering two half-space dielectrics in relative motion for example, the Poynting vector at the planar interface would give a measure of the energy exchange between the plates responsible for a decrease in relative motion that is theorised to occur. The Poynting vector (I.50) whose Hermitian operator form is given in terms of the operators of the field as

$$
\hat{\mathbf{S}}(\mathbf{x},t)=\frac{1}{2\mu_0}\left[\hat{\mathbf{E}}(\mathbf{x},t)\times\hat{\mathbf{B}}(\mathbf{x},t)-\hat{\mathbf{B}}(\mathbf{x},t)\times\hat{\mathbf{E}}(\mathbf{x},t)\right]. \tag{II.117}
$$

To keep things simple, we begin by deriving $\left\langle\hat{\mathbf{E}}\times\hat{\mathbf{B}}\right\rangle$ to which we need only add its complex conjugate to recover $\left\langle\hat{\mathbf{S}}\right\rangle$. Substituting the expansions of the operators in terms of the creation and annihilation operators (II.109) and (II.111), we have

$$
\begin{aligned}
\left\langle\hat{\mathbf{E}}(\mathbf{x},t)\times\hat{\mathbf{B}}(\mathbf{x},t)\right\rangle = \hbar\sum_{\lambda=1}^{2}\int\limits_{k_z>0}\frac{d^3\mathbf{k}}{(2\pi)^3}K\ \mathbb{E}_{\mathsf{L}}^{\lambda}(\mathbf{k},\mathbf{x})\times\mathbb{B}_{\mathsf{L}}^{\lambda\star}(\mathbf{k},\mathbf{x}) \\
+\hbar\sum_{\lambda=1}^{2}\int\limits_{K_z<0}\frac{d^3\mathbf{K}}{(2\pi)^3}K\ \mathbb{E}_{\mathsf{R}}^{\lambda}(\mathbf{K},\mathbf{x})\times\mathbb{B}_{\mathsf{R}}^{\lambda\star}(\mathbf{K},\mathbf{x}).
\end{aligned}
\tag{II.118}
$$

After inserting the expression for the expansion coefficients for each polarisation, the Poynting vector expectation value can be written in terms of the reflection and transmission coefficients as

$$\left\langle\hat{\mathbf{E}}(\mathbf{x},t)\times\hat{\mathbf{B}}(\mathbf{x},t)\right\rangle=\frac{\hbar}{2}\int\frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\limits_0^\infty\frac{dk_z}{2\pi}K\left[\left|t_L^1(k_z,K_z)\mathrm{e}^{\mathrm{i}K_zz}\right|^2\hat{\mathbf{K}}^\star+\left|t_L^2(k_z,K_z)\mathrm{e}^{\mathrm{i}K_zz}\right|^2\hat{\mathbf{K}}\right]$$
$$+\frac{\hbar}{2}\int\frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\limits_{-\infty}^0\frac{dK_z}{2\pi}\left[2+r_R^{1\star}(k_z,K_z)\mathrm{e}^{\mathrm{i}2K_zz}+r_R^{2\star}(k_z,K_z)\mathrm{e}^{\mathrm{i}2K_zz}\right]\mathbf{K}$$
$$+\left[r_R^1(k_z,K_z)\mathrm{e}^{-\mathrm{i}2K_zz}+r_R^2(k_z,K_z)\mathrm{e}^{-\mathrm{i}2K_zz}+\left|r_R^1(k_z,K_z)\right|^2+\left|r_R^2(k_z,K_z)\right|^2\right]\mathbf{K}^R. \tag{II.119}$$

Note that in the first line of this expression, since K_z may be complex, the exponential terms inside the absolute values do not always disappear. This is because for certain values of k_z, K_z may be complex. The reflection coefficients are even in $\mathbf{k}_{||}$ therefore the integrals in the direction of $\mathbf{k}_{||}$ go to zero leaving

$$\begin{aligned}&\left\langle\hat{\mathbf{E}}(\mathbf{x},t)\times\hat{\mathbf{B}}(\mathbf{x},t)\right\rangle\\&=\frac{\hbar}{2}\int\frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\limits_0^\infty\frac{dk_z}{2\pi}\left[\left|t_L^1\left(k_z,K_z\right)\mathrm{e}^{\mathrm{i}K_zz}\right|^2K_z^\star+\left|t_L^2\left(k_z,K_z\right)\mathrm{e}^{\mathrm{i}K_zz}\right|^2K_z\right]\hat{\mathbf{z}}\\&+\frac{\hbar}{2}\int\frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\limits_{-\infty}^0\frac{dK_z}{2\pi}\Bigg[2+\mathrm{Im}\left[r^{1\star}(k_z,K_z)\mathrm{e}^{\mathrm{i}2K_zz}\right]+\mathrm{Im}\left[r^{2\star}(k_z,K_z)\mathrm{e}^{\mathrm{i}2K_zz}\right]\\&-\left|r_R^1(k_z,K_z)\right|^2-\left|r_R^2(k_z,K_z)\right|^2\Bigg]K_z\hat{\mathbf{z}}.\end{aligned} \tag{II.120}$$

In the integral expression in the first line, K_z is complex below a certain critical value of k_z, namely $k_c=k_{||}\sqrt{\mathrm{n}^2-1}$. This allows us to separate this integral into two parts,

$$\begin{aligned}&\int\frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\limits_0^\infty\frac{dk_z}{2\pi}\left[\left|t_L^1\left(k_z,K_z\right)\mathrm{e}^{\mathrm{i}K_zz}\right|^2K_z^\star+\left|t_L^2\left(k_z,K_z\right)\mathrm{e}^{\mathrm{i}K_zz}\right|^2K_z\right]\hat{\mathbf{z}}\\&=\int\frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\limits_{k_c}^\infty\frac{dk_z}{2\pi}\left[\left|t_L^1\left(k_z,K_z\right)\right|^2K_z^\star+\left|t_L^2\left(k_z,K_z\right)\right|^2K_z\right]\hat{\mathbf{z}}\\&\qquad+i\int\frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\limits_0^{k_c}\frac{dk_z}{2\pi}\left[\left|t_L^1\left(k_z,K_z\right)\right|^2-\left|t_L^2\left(k_z,K_z\right)\right|^2\right]\zeta e^{-2\zeta z}\,\hat{\mathbf{z}}\end{aligned} \tag{II.121}$$

where in the last line, we have made the replacement $K_z \to i\zeta$. Now, note that using expressions (II.108), a change of variables from K_z to k_z results in the change of the integral

$$\int_{-\infty}^{0} dK_z \equiv \int_{-\infty}^{-k_c} dk_z \frac{\partial K_z}{\partial k_z} \tag{II.122}$$

and by manipulating the Fresnel coefficients (II.115), we find that

$$\left|t_L^{\lambda}\left(k_z, K_z\right)\right|^2 = \frac{\partial K_z}{\partial k_z}\left[1 - \left|r_R^{\lambda}\left(k_z, K_z\right)\right|^2\right] \tag{II.123}$$

where the partial derivative can be written as

$$\frac{\partial K_z}{\partial k_z} = \frac{k_z}{\mathsf{n}^2 K_z}. \tag{II.124}$$

Thus, the first line of (II.121), where K_z is real, cancels with parts of the second line of (II.120), i.e

$$\begin{aligned}\int \frac{d^2\mathbf{k}_{||}}{(2\pi)^2} \int_{k_c}^{\infty} \frac{dk_z}{2\pi}\left[\left|t_L^1\left(k_z, K_z\right)\right|^2 + \left|t_L^2\left(k_z, K_z\right)\right|^2\right] K_z\,\hat{\mathbf{z}} \\ + \int \frac{d^2\mathbf{k}_{||}}{(2\pi)^2} \int_{-\infty}^{-k_c} \frac{dk_z}{2\pi}\frac{\partial K_z}{\partial k_z}\left[2 - \left|r_R^1(k_z, K_z)\right|^2 - \left|r_R^2(k_z, K_z)\right|^2\right] K_z\hat{\mathbf{z}} = 0.\end{aligned} \tag{II.125}$$

This leaves the following expression for the vacuum expectation of the Poynting vector

$$\begin{aligned}\left\langle \hat{\mathbf{E}}(\mathbf{x},t) \times \hat{\mathbf{B}}(\mathbf{x},t)\right\rangle = \frac{i\hbar}{2}\int \frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int_0^{k_c}\frac{dk_z}{2\pi}\left[\left|t_L^2\left(k_z, K_z\right)\right|^2 - \left|t_L^1\left(k_z, K_z\right)\right|^2\right]\zeta e^{-2\zeta z}\,\hat{\mathbf{z}} \\ + \frac{\hbar}{2}\int \frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{dK_z}{2\pi}\left[r_R^1(k_z, K_z) + r_R^2(k_z, K_z)\right]\mathrm{e}^{\mathrm{i}2K_z z}K_z\,\hat{\mathbf{z}}.\end{aligned} \tag{II.126}$$

Consider for a moment, the first term of the integral in the second line,

$$I = \int \frac{d^2\mathbf{k}_{||}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dK_z}{2\pi}\, r_R^1(k_z, K_z)\, \mathrm{e}^{\mathrm{i}2K_z z} K_z\, \hat{\mathbf{z}}. \tag{II.127}$$

The expression for the reflection coefficient $r_R^1(k_z, K_z)$ from (II.115) written just in terms of K_z is

$$r_R^1(K_z) = \frac{K_z - \sqrt{n^2 K_z^2 + (\mathsf{n}^2 - 1)k_{||}^2}}{K_z + \sqrt{\mathsf{n}^2 K_z^2 + (\mathsf{n}^2 - 1)k_{||}^2}}. \tag{II.128}$$

Around a point where the square roots disappear, namely $K_z = \pm i\zeta_c + \delta \mathrm{e}^{\mathrm{i}\theta}$ where $\zeta_c = \mathsf{n}^{-1}\left|k_{||}\right|\sqrt{\mathsf{n}^2 - 1}$ and $\delta \mathrm{e}^{\mathrm{i}\theta}$ is some small complex number, this function can be approximated as

$$r_R^1(i\zeta_c + \delta \mathrm{e}^{\mathrm{i}\theta}) = \frac{i\zeta_c - \mathsf{n}\sqrt{2i\zeta_c\delta}\, \mathrm{e}^{\mathrm{i}\theta/2}}{i\zeta_c + \mathsf{n}\sqrt{2i\zeta_c\delta}\, \mathrm{e}^{\mathrm{i}\theta/2}}. \tag{II.129}$$

Now, if we imagine tracing out a contour around this point by varying θ from 0 to 2π, i.e essentially making the replacement $\theta \to \theta + 2\pi$, instead of getting back the same value, what we instead get is

$$r_R^1(i\zeta_c + \delta \mathrm{e}^{\mathrm{i}\theta}) = \frac{i\zeta_c + \mathsf{n}\sqrt{2i\zeta_c\delta}\, \mathrm{e}^{\mathrm{i}\theta/2}}{i\zeta_c - \mathsf{n}\sqrt{2i\zeta_c\delta}\, \mathrm{e}^{\mathrm{i}\theta/2}}. \tag{II.130}$$

The fact that $r_R^1(\theta) \neq r_R^1(\theta + 2\pi)$ shows that these points $\pm i\zeta_c$ are in fact *branch points*. Thus, we take the branch cut in this reflection coefficient to extend between $-i\zeta_c$ and $i\zeta_c$. Thus, analogously to [82], the integral (II.127) can be extended to the complex plane as seen in Figure II.9 and we integrate along the contour C around the branch cut passing vertically through the point where Re $[K_z] = 0$. The integral is composed of an integral from 0 to ∞ and an integral from $-\infty$ to 0 but *excluding* 0. This allows us to join the two integrals up using the big arc and the contour around the branch cut in Figure II.9. This leads to

$$
\begin{aligned}
&\oint \frac{dK_z}{2\pi} r_R^1(k_z, K_z) \mathrm{e}^{\mathrm{i}2K_z z} K_z \\
&= \int_{\delta}^{\infty} \frac{dK_z}{2\pi} r_R^1(k_z, K_z) \mathrm{e}^{\mathrm{i}2K_z z} K_z + \int_{-\infty}^{-\delta} \frac{dK_z}{2\pi} r_R^1(k_z, K_z) \mathrm{e}^{\mathrm{i}2K_z z} K_z \\
&+ \int_{K_z=-\delta}^{K_z=-\delta+i\zeta_c} \frac{dK_z}{2\pi} r_R^1(k_z, K_z) \mathrm{e}^{\mathrm{i}2K_z z} K_z + \int_{K_z=\delta+i\zeta_c}^{K_z=\delta} \frac{dK_z}{2\pi} r_R^1(k_z, K_z) \mathrm{e}^{\mathrm{i}2K_z z} K_z \\
&+ \delta^2 \int_{\pi}^{0} \frac{d\theta}{2\pi} r_R^1(k_z, \delta \mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{\mathrm{i}2\delta \mathrm{e}^{\mathrm{i}\theta} z} \mathrm{e}^{\mathrm{i}2\theta} + R^2 \int_{0}^{\pi} \frac{d\theta}{2\pi} r_R^1(k_z, R\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{\mathrm{i}2R\mathrm{e}^{\mathrm{i}\theta} z} \mathrm{e}^{\mathrm{i}2\theta} = 0.
\end{aligned}
\tag{II.131}
$$

In the limit $R \to \infty$ and $\delta \to 0$, the integrals over the two arcs go to zero. This

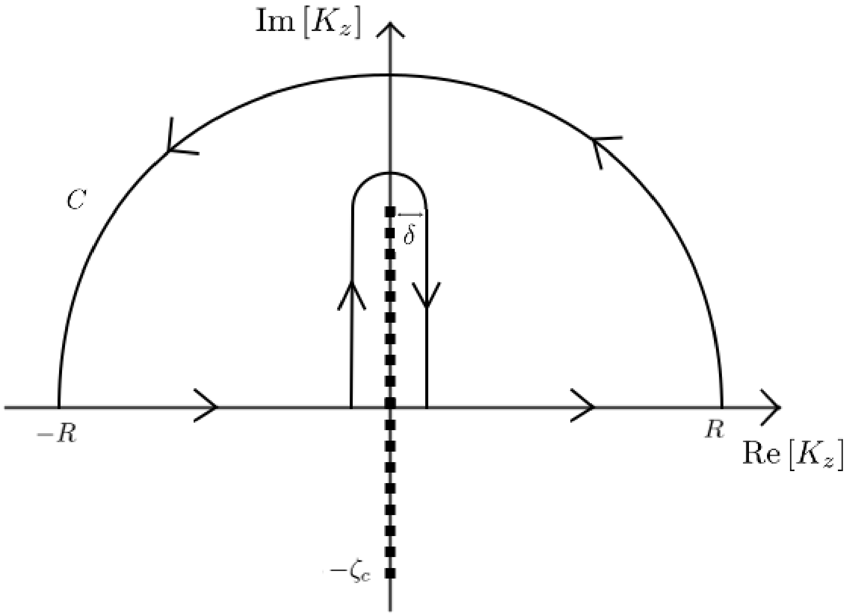

Figure II.9: Contour integration of $f(K_z) = r^1(K_z)\exp(i2K_z z)K_z$ along contour C consisting of an integral from along the real axis excluding $K_z = 0$, a a contour around the branch cut extending between $-\zeta_c$ and ζ_c in the limit $\delta \to 0$, and a counterclockwise arc at $|K_z| \to \infty$.

leads to the expression for the integral (II.127)

$$\begin{aligned} I &= \int_{K_z=0}^{K_z=i\zeta_c} \frac{dK_z}{2\pi} \left(\frac{K_z + k_z}{K_z - k_z} - \frac{K_z - k_z}{K_z + k_z} \right) \mathrm{e}^{\mathrm{i}2K_z z} K_z \\ &= i \int_{k=0}^{k=k_c} \frac{dk_z}{2\pi} \frac{k_z}{n^2 \zeta} \left(\frac{4\zeta k_z}{\zeta^2 + k_z^2} \right) e^{-2\zeta z} \zeta \end{aligned} \tag{II.132}$$

where we performed the change of variables $K_z \to \mathrm{i}\zeta$. We also used the fact that $k_c = \mathrm{i}n\zeta_c$. Again, using the expressions for the reflection and transmission coefficients (II.115), this is equivalent to

$$I = i \int_{k=0}^{k=k_c} \frac{dk_z}{2\pi} \left| t_1 \left(k_z, \zeta \right) \right|^2 e^{-2\zeta z} \zeta. \tag{II.133}$$

Using this result for the quantity I, we can use (II.127) to show that the first integral of the second line in (II.126) cancels out the second integral of the first line in (II.126). Thus the contribution of polarization 1 to expression (II.118) is zero. An identical argument also holds for the second polarization (after all the decomposition into the two polarizations is arbitrary), and we thus recover the anticipated result that the vacuum expectation value of the Poynting vector near the surface of a stationary, isotropic dielectric is zero. We now wish to use the same formalism to understand how the motion of the medium effects the energy near the surface.

II.3.2 Moving Media

We now consider the case where the half space dielectric is moving at a constant velocity $\mathbf{v}$ in the $\hat{\mathbf{x}}$-direction with speed v as depicted in Figure II.8. The Poynting vector in the co-moving frame is denoted $\left\langle \hat{\mathbf{S}}' \left(\mathbf{x}', t' \right) \right\rangle$ where $\mathbf{x}'$ and t' are the Lorentz boosted coordinates. It can be written analogously to (II.117) in terms of the co-moving electric and magnetic fields via

$$\left\langle \hat{\mathbf{S}}'(\mathbf{x}', t') \right\rangle = \frac{1}{2\mu_0} \left[\left\langle \hat{\mathbf{E}}'(\mathbf{x}', t') \times \hat{\mathbf{B}}'(\mathbf{x}', t') \right\rangle - \left\langle \hat{\mathbf{B}}'(\mathbf{x}', t') \times \hat{\mathbf{E}}'(\mathbf{x}', t') \right\rangle \right]. \tag{II.134}$$

The co-moving electric and magnetic field operators are expanded in terms of the creation and annihilation operators from the *lab* frame (in which the medium is in motion). The electric field operator is expanded as

$$\hat{\mathbf{E}}'(\mathbf{x}',t') = \sum_{\lambda=1}^{2} \int\limits_{k_z>0} \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{cK}\left[\mathbb{E}_L^{\lambda}(\mathbf{k}',\mathbf{x}')\,\mathrm{e}^{-\mathrm{i}\omega(K')t'}\hat{u}_{\lambda}(\mathbf{k}) + \text{h.c}\right] + \sum_{\lambda=1}^{2} \int\limits_{K_z<0} \frac{d^3\mathbf{K}}{(2\pi)^3}\sqrt{cK}\left[\mathbb{E}_R^{\lambda}(\mathbf{K}',\mathbf{x}')\,\mathrm{e}^{-\mathrm{i}\omega(K')t'}\hat{v}_{\lambda}(\mathbf{K}) + \text{h.c}\right] \tag{II.135}$$

and similarly the magnetic field operator by

$$\hat{\mathbf{B}}'(\mathbf{x}',t') = \sum_{\lambda=1}^{2} \int\limits_{k_z>0} \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{\frac{K}{c}}\left[\mathbb{B}_L^{\lambda}(\mathbf{k}',\mathbf{x}')\,\mathrm{e}^{-\mathrm{i}\omega(K')t'}\hat{u}_{\lambda}(\mathbf{k}) + \text{h.c}\right] + \sum_{\lambda=1}^{2} \int\limits_{K_z<0} \frac{d^3\mathbf{K}}{(2\pi)^3}\sqrt{\frac{K}{c}}\left[\mathbb{B}_R^{\lambda}(\mathbf{K}',\mathbf{x}')\,\mathrm{e}^{-\mathrm{i}\omega(K')t'}\hat{v}_{\lambda}(\mathbf{K}) + \text{h.c}\right] \tag{II.136}$$

respectively. Note that the creation and annihilation operators are functions of the lab frame wave-vectors $\mathbf{K}$ and $\mathbf{k}$ not $\mathbf{K}'$ and $\mathbf{k}'$. Had they been expanded in terms of the co-moving creation and annihilation operators, the vacuum expectation of the Poynting vector would just be zero as in the stationary case. The coefficients are given by analogous expressions to (II.112) and (II.113). The frequency and wave-vectors transform according to

$$\begin{aligned} K' &= \gamma\,(K - \beta K_x) \\ k_x' &= \gamma\,(k_x - \beta k) \\ K_y' &= K_y \\ k_y' &= k_y \end{aligned} \tag{II.137}$$

with the the z components given by

$$
\begin{aligned}
K'_z &= \pm\sqrt{K'^2 - k'^2_x - k^2_y} \\
k'_z &= \pm\sqrt{k'^2 - k'^2_x - k^2_y}.
\end{aligned}
\tag{II.138}
$$

Substituting the expansions of the operators (II.135) and (II.136) into (II.134), and applying the commutation relations (II.110), the Poynting vector in the co-moving frame can be expressed as the integral

$$
\begin{aligned}
\left\langle \hat{\mathbf{S}}'(\mathbf{x}', t') \right\rangle = \hbar \sum_{\lambda=1}^{2} \int\limits_{k_z>0} \frac{d^3\mathbf{k}}{(2\pi)^3} K \left[\mathbb{E}^{\lambda}_{L}(\mathbf{k}', \mathbf{x}') \times \mathbb{B}^{\lambda\star}_{L}(\mathbf{k}', \mathbf{x}') + \text{c.c}\right] \\
+ \hbar \sum_{\lambda=1}^{2} \int\limits_{K_z<0} \frac{d^3\mathbf{K}}{(2\pi)^3} K \left[\mathbb{E}^{\lambda}_{R}(\mathbf{K}', \mathbf{x}') \times \mathbb{B}^{\lambda\star}_{R}(\mathbf{K}', \mathbf{x}') + \text{c.c}\right].
\end{aligned}
\tag{II.139}
$$

Ultimately, we want to calculate the flow of energy along the surface of the medium as it moves through the vacuum, this is equivalent to energy being dragged along with the medium in the lab frame. The Lorentz boosted coefficients $\mathbb{E}^{\lambda}(\mathbf{k}', \mathbf{x}')$ and $\mathbb{B}^{\lambda}(\mathbf{k}', \mathbf{x}')$, can be written in terms of the rest frame coefficients by considering the general rule for the transformation of fields (I.7) and (I.8). This allows us to write the unit polarisation vectors are expressed in terms of the rest frame vectors via

$$
\begin{aligned}
\mathbf{e}^1(\mathbf{k}') &= \mathbf{e}^1_{||}(\mathbf{k}) + \gamma\left[\mathbf{e}^1_{\perp}(\mathbf{k}) + \mathbf{v} \times \mathbf{e}^2(\mathbf{k})\right] \\
\mathbf{e}^2(\mathbf{k}') &= \mathbf{e}^2_{||}(\mathbf{k}) + \gamma\left[\mathbf{e}^2_{\perp}(\mathbf{k}) - \mathbf{v} \times \mathbf{e}^1(\mathbf{k})\right].
\end{aligned}
\tag{II.140}
$$

From these, we can establish expression for the integrands in (II.139) in terms of the rest frame coefficients, we give as an example one of the terms for the transverse electric polarisation

$$\begin{aligned}\mathbb{E}_L^1\left(\mathbf{k}',\mathbf{x}'\right)\times\mathbb{B}_L^{1\star}\left(\mathbf{k}',\mathbf{x}'\right)=\frac{1}{2}\left|t_L^1\left(k_z,K_z\right)e^{i\mathbf{K}'\cdot\mathbf{x}'}\right|^2\times\\ \gamma^2\left[\left(1+v^2\right)\frac{K_x}{K}-v\left(\frac{K_{||}^2K_{||}^2+K_x^2K_z^2+K_y^2\left|K_z\right|^2}{K^2K_{||}^2}+\frac{K_x^2}{K^2}\right)\right]\hat{\mathbf{x}}\\ +\gamma\left[\frac{K_y}{K}-v\frac{K_xK_y}{K^2}\left(\frac{K^2-\left|K_z\right|^2}{K_{||}^2}\right)\right]\hat{\mathbf{y}}+\gamma\left[\frac{K_z^\star}{K}-v\frac{K_xK_z^\star}{K^2}\right]\hat{\mathbf{z}}\end{aligned} \tag{II.141}$$

with similar expressions for the other polarisations. From the integral expression for the full Poynting vector (II.139), one can see that eventually these will be integrated over all directions of $\mathbf{k}$ and $\mathbf{K}$, but for both parts of the expression, the integrals over k_x, k_y, K_x and K_y extend from $-\infty$ to ∞, thus any terms that are odd in these variables will disappear. We can therefore omit the terms that integrate to zero in (II.141) leaving

$$\begin{aligned}\mathbb{E}_L^1\left(\mathbf{k}',\mathbf{x}'\right)\times\mathbb{B}_L^{1\star}\left(\mathbf{k}',\mathbf{x}'\right)\equiv\frac{1}{2}\left|t_L^1\left(k_z,K_z\right)e^{i\mathbf{K}'\cdot\mathbf{x}'}\right|^2\times\\ \left[-v\gamma^2\left(\frac{K_{||}^4+K_x^2K_z^2+K_y^2\left|K_z\right|^2}{K^2K_{||}^2}+\frac{K_x^2}{K^2}\right)\hat{\mathbf{x}}+\gamma\frac{K_z^\star}{K}\hat{\mathbf{z}}\right]\end{aligned} \tag{II.142}$$

where, again, we have similar expressions for the other polarisation (the terms with a ‘R’ subscript turn out to be considerably more complicated due to them containing both incident and reflected terms). Since in reality we will integrate over the real parts of these, adding the complex conjugates to all the terms gives

$$\begin{aligned}\mathbb{E}_L^1\left(\mathbf{k}',\mathbf{x}'\right)\times\mathbb{B}_L^{1\star}\left(\mathbf{k}',\mathbf{x}'\right)+\text{c.c}\equiv\left|t_L^1\left(k_z,K_z\right)e^{i\mathbf{K}'\cdot\mathbf{x}'}\right|^2\times\\ \left[-v\gamma^2\left(\frac{K_{||}^4+K_x^2\text{Re}\left[K_z^2\right]+K_y^2\left|K_z\right|^2}{K^2K_{||}^2}+\frac{K_x^2}{K^2}\right)\hat{\mathbf{x}}+\gamma\frac{\text{Re}\left[K_z\right]}{K}\hat{\mathbf{z}}\right].\end{aligned} \tag{II.143}$$

Now, if we sum over the two polarisations, for the terms corresponding to waves originating from within the medium give

$$\sum_{\lambda}\left[\mathbb{E}_L^{\lambda}\left(\mathbf{k}',\mathbf{x}'\right)\times\mathbb{B}_L^{\lambda\star}\left(\mathbf{k}',\mathbf{x}'\right)+\text{c.c}\right]=\left[\left|t_L^1\left(k_z,K_z\right)e^{i\mathbf{K}'\cdot\mathbf{x}'}\right|^2+\left|t_L^2\left(k_z,K_z\right)e^{i\mathbf{K}'\cdot\mathbf{x}'}\right|^2\right]$$
$$\times\left[-v\gamma^2\left(\frac{K_{||}^4+K_x^2\text{Re}\left[K_z^2\right]+K_y^2\left|K_z\right|^2}{K^2K_{||}^2}+\frac{K_x^2}{K^2}\right)\hat{\mathbf{x}}+\gamma\frac{\text{Re}\left[K_z\right]}{K}\hat{\mathbf{z}}\right] \tag{II.144}$$

and a more intricate expression for the terms originating from the vacuum

$$\sum_{\lambda}\left[\mathbb{E}_R^{\lambda}\left(\mathbf{k}',\mathbf{x}'\right)\times\mathbb{B}_R^{\lambda\star}\left(\mathbf{k}',\mathbf{x}'\right)+\text{c.c}\right]=2\left[-\gamma^2v\left(1+\frac{K_x^2}{K^2}\right)\hat{\mathbf{x}}+\gamma\frac{K_z}{K}\hat{\mathbf{z}}\right]$$
$$-\gamma^2v\left(\frac{K_{||}^4+K_x^2K_z^2-K_y^2K_z^2}{K^2K_{||}^2}+\frac{K_x^2}{K^2}\right)\left[r_R^1\left(k_z,K_z\right)+r_R^2\left(k_z,K_z\right)\right]e^{-i2K_zz}\,\hat{\mathbf{x}}$$
$$-\gamma^2v\left(\frac{K_{||}^4+K_x^2K_z^2-K_y^2K_z^2}{K^2K_{||}^2}+\frac{K_x^2}{K^2}\right)\left[r_R^{1\star}\left(k_z,K_z\right)+r_R^{2\star}\left(k_z,K_z\right)\right]e^{i2K_zz}\,\hat{\mathbf{x}}$$
$$-\left[\gamma^2v\left(1+\frac{K_x^2}{K^2}\right)\hat{\mathbf{x}}+\gamma\frac{K_z}{K}\hat{\mathbf{z}}\right]\left[\left|r_R^1\left(k_z,K_z\right)\right|^2+\left|r_R^2\left(k_z,K_z\right)\right|^2\right]. \tag{II.145}$$

We can immediately see that the integrals over the components in the z direction will cancel out in the exact same way as in (II.125). Now, for the remaining parts, which are all in the x direction, i.e the direction of motion, the exponential terms can be written as

$$e^{i\mathbf{K}'\cdot\mathbf{x}'}=e^{i\mathbf{K}_\perp\cdot\mathbf{x}_\perp}e^{iK_x'x_x'}=e^{i\mathbf{K}_\perp\cdot\mathbf{x}_\perp}e^{i\gamma^2\left[K(x-vt)-\frac{v\omega}{c^2}(x-vt)\right]} \tag{II.146}$$

the absolute value of which either becomes $\left|e^{i\mathbf{K}'\cdot\mathbf{x}'}\right|^2=1$ when the K_z is real, or $\left|e^{i\mathbf{K}'\cdot\mathbf{x}'}\right|^2=e^{-2\zeta z}$ when K_z is complex. To elucidate this further, we can write (II.143) in the form it takes when K_z is real, which is

$$\mathbb{E}_L^1\left(\mathbf{k}',\mathbf{x}'\right)\times\mathbb{B}_L^{1\star}\left(\mathbf{k}',\mathbf{x}'\right)+\text{c.c}=-v\gamma^2\left(1+\frac{K_x^2}{K^2}\right)\left[\left|t_L^1\left(k_z,K_z\right)\right|^2+\left|t_L^2\left(k_z,K_z\right)\right|^2\right]\hat{\mathbf{x}} \tag{II.147}$$

and the form it takes when K_z is imaginary, where we make the replacement $K_z \to \mathrm{i}\zeta_z$

$$\mathbb{E}_L^1(\mathbf{k}',\mathbf{x}') \times \mathbb{B}_L^{1\star}(\mathbf{k}',\mathbf{x}') + \text{c.c} = -v\gamma^2 \left(\frac{K_{||}^2 K_{||}^2 + K_x^2 K_{||}^2 + K_y^2\zeta^2}{K^2 K_{||}^2}\right) \times \left[\left|t_L^1(k_z, i\zeta)\right|^2 + \left|t_L^2(k_z, i\zeta)\right|^2\right] e^{-2\zeta z}\,\hat{\mathbf{x}}. \tag{II.148}$$

Inserting expressions (II.144) and (II.145) into (II.139) and simplifying, we can see that the Poynting vector is equivalent to

$$\left\langle \hat{\mathbf{S}}'(\mathbf{x}',t')\right\rangle = -\frac{4}{3}\beta\gamma^2 \left\langle\hat{\rho}\right\rangle. \tag{II.149}$$

To understand this result, it is useful to consider the vacuum case. Using the expansion of the electric and magnetic field operators (I.40) and (I.41), the stationary Poynting Vector can again be calculated via (II.117) where

$$\left\langle \hat{\mathbf{E}}(\mathbf{x},t) \times \hat{\mathbf{B}}(\mathbf{x},t)\right\rangle = \frac{\hbar}{2}\sum_{\lambda,\lambda'}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d^3\mathbf{k}'}{(2\pi)^3}\sqrt{\frac{\omega(\mathbf{k})}{\omega(\mathbf{k}')}} \times \langle 0| \left[\mathbf{e}^{\lambda}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}[\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t]}\hat{a}_{\lambda}(\mathbf{k}) + \text{h.c}\right] \times \left[\mathbf{k}' \times \mathbf{e}^{\lambda'}(\mathbf{k}')\, \mathrm{e}^{\mathrm{i}[\mathbf{k}'\cdot\mathbf{x}-\omega(\mathbf{k}')t]}\hat{a}_{\lambda'}(\mathbf{k}') + \text{h.c}\right]|0\rangle \tag{II.150}$$

which, once again applying the commutation relations (I.42) gives

$$\left\langle \hat{\mathbf{S}}(\mathbf{x},t)\right\rangle \equiv \frac{\hbar}{2}\sum_{\lambda}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\mathbf{e}^{\lambda}(\mathbf{k}) \times \left[\mathbf{k} \times \mathbf{e}^{\lambda}(\mathbf{k})\right]. \tag{II.151}$$

This can be simplified by applying the following vector identity

$$\mathbf{e}^{\lambda}(\mathbf{k}) \times \left[\mathbf{k} \times \mathbf{e}^{\lambda}(\mathbf{k})\right] = \mathbf{k}\left[\mathbf{e}^{\lambda}(\mathbf{k}) \cdot \mathbf{e}^{\lambda}(\mathbf{k})\right] - \mathbf{e}^{\lambda}(\mathbf{k})\left[\mathbf{k} \cdot \mathbf{e}^{\lambda}(\mathbf{k})\right] \tag{II.152}$$

which, given that by definition, $\mathbf{k} \cdot \mathbf{e}^{\lambda}(\mathbf{k}) = 0$ and $\mathbf{e}^{\lambda}(\mathbf{k}) \cdot \mathbf{e}^{\lambda}(\mathbf{k}) = 1$ results in a vanishing vacuum Poynting vector as one would expect

$$\left\langle \hat{\mathbf{S}}(\mathbf{x},t)\right\rangle = \hbar\int \frac{d^3\mathbf{k}}{(2\pi)^3}\mathbf{k} = 0. \tag{II.153}$$

However, if we now move to another reference frame moving, like in the previous section, with a relative velocity $\mathbf{v} = v\hat{\mathbf{x}}$, using the electric field operator expression

$$\hat{\mathbf{E}}'(\mathbf{x}', t') = \sum_{\lambda} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{\frac{\hbar\omega(\mathbf{k})}{2}} \left[\mathbf{e}^{\lambda}(\mathbf{k}')\, \mathrm{e}^{\mathrm{i}[\mathbf{k}'\cdot\mathbf{x}' - \omega(\mathbf{k}')t']} \hat{a}_{\lambda}(\mathbf{k}) + \text{h.c}\right] \tag{II.154}$$

and the magnetic field operator

$$\hat{\mathbf{B}}'(\mathbf{x}', t') = \sum_{\lambda} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{\frac{\hbar\omega(\mathbf{k})}{2}} \left[\hat{\mathbf{k}}' \times \mathbf{e}^{\lambda}(\mathbf{k}')\, \mathrm{e}^{\mathrm{i}[\mathbf{k}'\cdot\mathbf{x}' - \omega(\mathbf{k}')t']} \hat{a}_{\lambda}(\mathbf{k}) + \text{h.c}\right] \tag{II.155}$$

where the unit vectors can again be written in terms of the rest frame unit vectors (II.140). Again, the Poynting can be calculated and can be written in terms of the rest frame unit vectors as

$$\begin{aligned} \left\langle \hat{\mathbf{S}}(\mathbf{x}', t') \right\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \hbar\omega(\mathbf{k})\, \gamma^2 \left[\left(1 + v^2\right) \frac{k_x}{k} - v\left(1 + \frac{k_x^2}{k^2}\right)\right] \hat{\mathbf{x}} \\ + \gamma\left(v\frac{k_x k_y}{k^2} + \frac{k_y}{k}\right) \hat{\mathbf{y}} + \gamma\left(\frac{k_z}{k} - v\frac{k_x k_z}{k^2}\right) \hat{\mathbf{z}}. \end{aligned} \tag{II.156}$$

Both the y and z components of the Poynting vector only contain odd functions of the components of the wave-vector $\mathbf{k}$ and integrate to zero, as does the first part of the component in the direction x. The remaining terms give

$$\left\langle \hat{\mathbf{S}}(\mathbf{x}', t') \right\rangle = -v\gamma^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \hbar\omega(\mathbf{k}) \left(1 + \frac{k_x^2}{k^2}\right) \hat{\mathbf{x}} \tag{II.157}$$

which can be seen to equivalent to

$$\left\langle \hat{\mathbf{S}}(\mathbf{x}', t') \right\rangle = -\frac{4}{3} v\gamma^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \hbar\omega(\mathbf{k})\, \hat{\mathbf{x}} \tag{II.158}$$

or, written in terms of the vacuum energy density

$$\left\langle \hat{\mathbf{S}}(\mathbf{x}', t') \right\rangle = -\frac{4}{3}\beta\gamma^2 \left\langle \hat{\rho} \right\rangle \hat{\mathbf{x}} \tag{II.159}$$

where the energy density is given as

$$\langle\hat{\rho}\rangle = \frac{1}{2}\int d^3\mathbf{k}\,\hbar\omega\left(\mathbf{k}\right). \tag{II.160}$$

This means, that even in the vacuum, moving to a moving frame of reference results in an infinite Poynting vector in the opposite direction to the motion, almost as though the energy has a preferred frame of reference relative to which our frame is now moving, similar to an ether perhaps. However, this is not paradoxical. Indeed in this vacuum case, we could have come to this result without having ever considered the expansions (II.154) and (II.155). One simply needs to consider a Lorentz transformation of the Energy-Stress tensor (I.48) to see that the x component of the Poynting vector is

$$\begin{aligned}\left\langle\bar{T}^{01}\right\rangle &= -\beta\gamma^2\left[\epsilon_0\left(\left\langle E^2\right\rangle - \left\langle E_{||}^2\right\rangle\right) + \frac{1}{\mu_0}\left(\left\langle B^2\right\rangle - \left\langle B_{||}^2\right\rangle\right)\right] \\ &= -\frac{2}{3}\beta\gamma^2\left[\epsilon_0\left\langle E^2\right\rangle + \frac{1}{\mu_0}\left\langle B^2\right\rangle\right] = -\frac{4}{3}\beta\gamma^2\left\langle T_{00}\right\rangle\end{aligned} \tag{II.161}$$

where $E_{||}$ and $B_{||}$ are defined as the components of the electric and magnetic fields parallel to the direction of motion. But what does this mean? The key is to return to the definition of the energy-stress tensor (I.48). It is a conserved quantity that must obey $\partial_\mu T^{\mu\nu} = 0$. Therefore, we can add to it any tensor $U^{\mu\nu\rho}$ so long as it satisfies $\partial_\rho U^{\mu\nu\rho} = 0$. Essentially, so long as this infinite energy does not depend on space or time in any special way, it can be neglected as it bears no influence on the equations of motion. Of course, for the vacuum energy density, this condition is automatically fulfilled since it must be homogeneous by definition (in the absence of matter). As such, we can arbitrarily add multiples of the vacuum energy to the energy momentum tensor without incurring any additional physical effects. Thus this infinite Poynting vector arising from travelling through the vacuum can simply be subtracted, recovering the stationary vacuum as one would expect.

The same result holds for our Poynting vector near the moving dielectric; since the direction of motion is parallel to the inhomogeneity (the boundary with the dielectric), the form it takes satisfies the condition $\partial_\mu U^{\mu\nu\rho} = 0$ that allows us to remove it by subtracting the appropriate cancelling term from the Poynting vector, leaving that there is no net energy flow along the moving surface per se. The conclusion to be drawn here is that the intuitive picture of the vacuum energy being dragged along with the medium is a subtle one and merely considering the Poynting vector is insufficient to illustrate this intuition in this case.

However, it seems that the reason for the transfer of a photon between the moving medium and the two level system in the previous sections rests on the fact that the presence of the two-level system breaks the symmetry in the direction of motion. The result is that one can no longer remove the energy flowing along the surface. The effect of the non-zero Poynting vector is to promote the exchange of energy between the two entities. It seems that this understanding must hold true for both the half space dielectric and the infintie medium of sections II.1 and II.2. However, note that this is only true for constant velocities. As discussed in the introduction, when the dielectric exhibits time dependent motion, such as in the Unruh effect [25], even without the presence of another body, the dielectric may exhibit a 'radiation damping' of sorts. In the next chapter, we will consider these effects in some detail using the more general case of media that dissipate and disperse the electromagnetic energy from the vacuum.

Chapter III

Macroscopic Quantum Electrodynamics and Moving Media

In the previous sections, we introduced the idea that the electromagnetic vacuum energy, while infinite, is modified by the presence of media and that these modifications can in turn lead to changes in the dynamics of those objects bathed within it. In Section II.2, we explored the absorption of a photon by a two level system placed inside a moving dielectric. We saw that while dielectric materials are often represented by an effective bulk property known as the refractive index n, or the related permittivity $\epsilon = n^2$, the origins of the forces from the quantum vacuum is at a scale on a par with the microscopic constituents of the media. Taking this microscopic structure into account sheds some light on the mechanisms at work by showing that the mysterious 'negative frequency' modes that appear in such calculations are due to the scale: at the mciroscopic scale these modes are absent but arise when the scale of the field is much larger than the distance between the particles making up the matter.

In Section II.3, the energy from the quantum vacuum surrounding a moving dielectric was also considered, and we explored the idea that moving media appears to 'drag' the vacuum along with it. If the vacuum outside the medium is homogeneous in the direction of motion, this vacuum energy can effectively be removed and has no physical effect. However, when the symmetries of the system are broken by the presence of a second particle such in Section II.2, certain modes of vacuum energy that are dragged along with the media have a real physical effect and serve to transfer energy from one body to the other.

We briefly mentioned in this section that in reality, no material has a purely real refractive index and only usually has an approximately real refractive over some finite range of frequencies. Therefore, for many real world problems, the dissipation and dispersion of the electromagnetic energy within matter must be included. Here, we will explore these effects in more detail. We will be working within a relatively new model for the interaction of lossy, dispersive dielectrics with quantum light [84], a framework that again approximates the interaction of light with media by a dielectric function that obeys the Kramer-Kronig relations (I.17) which relates the real and imaginary parts of this dielectric function, i.e. relates the dissipation and dispersion of light. We will again see the appearance of negative frequency modes [44].

First, this framework will be introduced and we will explain how the dissipation of electromagnetic energy into the dielectric is accounted for within the context of quantum field theory. Our purpose here, however, is to consider the effects that occur within the context of moving media, or more generally, time-dependent media (this also includes other time dependencies such as changing permittivities for example). After discussing some simple cases, we will explore the first order effects that occur for arbitrary time dependencies using the canonical variable approach to quantum field theory.

To begin with, let us consider for a moment the ability of media to absorb and disperse electromagnetic energy. Classically, of course, this is simply done by extending the permittivity and permeability constants to become complex functions of frequency where the imaginary part of this function serves to generate the appropriate damping of the fields. The can be seen in simple terms by considering the solution to a simple scalar wave equation (II.2) (where the permittivity $\epsilon = n^2$) which, in the frequency domain would be given by

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\epsilon(\omega)\right]\varphi(x,\omega) = 0. \tag{III.1}$$

One solution could be $\varphi(x,\omega) = \varphi_0 \exp(i\omega x\sqrt{\epsilon(\omega)}/c)$. If the permittivity function $\epsilon(\omega)$ is complex, then the exponent of this solution can be written as a complex number $(\alpha + \mathrm{i}\beta)x$ and this solution can be written in the form $\varphi(x,\omega) = \varphi_0 \exp(\alpha x)\exp(i\beta x)$. Clearly the first exponential serves to either damp or amplify the mode depending on the sign of α, which will correspond to either loss or gain of electromagnetic energy. The frequency dependence of these exponents is responsible for the dispersion of different modes of light. It is possible to include these effects separately, but in reality, they are in fact somewhat interdependent as we shall see later on.

Unfortunately however, when one is considering the effects of the quantum vacuum, the dissipation of energy from the electromagnetic field leads to difficulties in applying some of the usual tools at one's disposal in quantum theory. As we saw in Section II.3, quantising evanescent waves is a tricky process and evidently from the modes discussed above, any absorbing system will contain similar decaying modes. The key in being able to quantise such a system seems to reside in the conservation of energy; by including somewhere for this decaying energy to go, the quantisation procedure will be possible. Physically, the energy absorbed by the media will remain in the media, in the form of vibrational modes or excited states of the constituent particles, until it is eventually re-emitted as radiation. Mathematically, providing a rudimentary model for the microscopic degrees of freedom of the medium is the most effective approach to date. Such a model has been years in the making.

A very brief overview of its development is given to allow the reader a birds-eye view of its emergence and the current extensions of the theory to include such cases as chiral media [85, 86] and constant motion [44, 87]. We will then consider how the vacuum state of the electromagnetic field is modified by general time-dependencies using the perturbative techniques afforded by quantum field theory as well as the resultant modifications to the correlation functions between various field amplitudes in both the vacuum and at equilibrium finite temperature systems.

III.1 The Huttner and Barnett Model

In magneto-dielectric media, the response functions of the media to electromagnetic perturbations, the aforementioned permittivity ϵ and permeability μ, obey the Kramers-Kronig relations (I.17). The fact that the frequency dependence of the real part of the response function (responsible for the dispersion of electromagnetic energy) dictates the form of the imaginary part (responsible for the absorption of electromagnetic energy) and vice versa indicates that dissipation and dispersion are intricately linked. Any model capable of quantising the electromagnetic field interacting with dissipative, dispersing media must incorporate this relationship into its framework.

The first model to achieve this was by Huttner and Barnett in 1992 [88]. Their model is based on the Hopfield model [89] where the system is described by a field $\mathbf{X}$ representing the polarisation of the dielectric coupled to the electromagnetic field. This polarisation field is then coupled to a reservoir of oscillators $\mathbf{Y}_\omega$ for every natural frequency ω representing the degrees of freedom of the media. The collective excitation of the degrees of freedom of the medium and the EM field are quasi-particles named 'polaritons' (we shall consider these in more details presently). The Lagrangian density characterising the model is defined as the sum of the free Lagrangian density (I.34), the polarisation field $\mathscr{L}_{\text{Matter}}$, the oscillator bath $\mathscr{L}_{\text{Osc}}$ and the coupling between the fields $\mathscr{L}_{\text{Int}}$,

$$\mathscr{L} = \mathscr{L}_{\text{EM}} + \mathscr{L}_{\text{Matter}} + \mathscr{L}_{\text{Osc}} + \mathscr{L}_{\text{Int}}. \tag{III.2}$$

The polarisation is modelled by a field of harmonic oscillators $\mathbf{X}$ of resonant frequency ω_0 whose Lagrangian density is

$$\mathscr{L}_{\text{Matter}} = \frac{\rho}{2}\left(\dot{\mathbf{X}}^2 - \omega_0^2\mathbf{X}^2\right) \tag{III.3}$$

with a constant ρ and the reservoir into which flows the absorbed energy of the material is made up of a field of harmonic oscillators $\mathbf{Y}_\omega$ for every frequency ω,

$$\mathscr{L}_{\text{Osc}} = \frac{\rho}{2} \int_0^\infty d\omega \left(\dot{\mathbf{Y}}_\omega^2 - \omega^2 \mathbf{Y}_\omega^2 \right). \tag{III.4}$$

The interaction between the EM field, the matter polarisation field and the reservoir is given by

$$\mathscr{L}_{\text{Int}} = -\alpha \left(\mathbf{A} \cdot \dot{\mathbf{X}} + \phi \, \nabla \cdot \mathbf{X} \right) - \int_0^\infty d\omega \; v(\omega) \; \mathbf{X} \cdot \dot{\mathbf{Y}}_\omega \tag{III.5}$$

where α is a coupling constant that couples the EM field to the polarisation field and $v(\omega)$ is a coupling constant that couples the polarisation field to the bath of oscillators. The details of the quantisation will not be discussed here as they bear many resemblances to the following model which as well as being the model upon which our results are based, has the additional invitation of being greatly simplified and extents to a larger class of media. However, we simply summarise their original results that this model yields the following diagonalised Hamiltonian

$$\hat{H} = \int_{-\infty}^{\infty} d^3\mathbf{k} \int_0^\infty d\omega \; \hbar\omega \; \hat{C}^\dagger \left(\mathbf{k}, \omega \right) \hat{C} \left(\mathbf{k}, \omega \right) \tag{III.6}$$

where the $\hat{C}$ operators represent the polariton creation and annihilation operators, the quasi-particles touched upon during the introduction to this model. As previously stated, they represent a mixed excitation of the electromagnetic field and the matter degrees of freedom. They satisfy the bosonic commutation relations which are analogues of (I.42)

$$\left[\hat{C} \left(\mathbf{k}, \omega \right), \hat{C}^\dagger \left(\mathbf{k}', \omega' \right) \right] = \delta \left(\mathbf{k} - \mathbf{k}' \right) \delta \left(\omega - \omega' \right). \tag{III.7}$$

As the reader may notice, the polaritons are not confined to a fixed dispersion curve but instead occupy the whole $(\mathbf{k}, \omega)$-space; this appears to be a generic feature in models that quantised the EM field coupled to dissipative dielectric media and one that is in common with the full theory of macroscopic QED that we cover in the next section. The field operators are expanded in terms of these polariton creation and annihilation operators. For example the electric field is expanded via

$$\begin{aligned}\mathbf{E}(x,t) = i\sqrt{\frac{\hbar\omega_c^2}{2\epsilon_0}}\int_{-\infty}^{\infty}\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\sum_{\lambda=1,2}\mathbf{e}_\lambda(\mathbf{k}) \\ \times\int_0^{\infty}d\omega\left[\frac{\omega\zeta^*(\omega)}{\omega^2\epsilon(\omega)-k^2c^2}\hat{C}(\mathbf{k},\omega)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} - \text{h.c}\right]\end{aligned} \tag{III.8}$$

where $\zeta(\omega)$ is a complex constant such that $\xi(\omega) = |\zeta(\omega)|^2/\omega$ and $\epsilon(\omega)$ defined by

$$\epsilon(\omega) = 1 + \frac{\omega_c^2}{2\omega}\left[\mathsf{P}\int_{-\infty}^{\infty}d\omega'\frac{\xi(\omega')}{\omega'-\omega} + i\pi\xi(\omega)\right] \tag{III.9}$$

is the usual complex dielectric constant satisfying the Kramers-Kronig relations.

The strengths of this model lie in its ability to recover a dielectric constant that satisfies the Kramers-Kronig relations (I.17) while constructing a Hamiltonian that can be fully quantised allowing the application of the full tool set of quantum field theory to systems involving macroscopic dielectrics subject to interactions with quantum mechanical light. The model described is only valid for isotropic dielectrics but has since been extended to model inhomogeneous and magneto-dielectric media. As we shall now see, this model can be simplified down to a more user-friendly form known as Macroscopic Quantum Electrodynamics.

III.2 Macroscopic Quantum Electrodynamics

Huttner and Barnett's model was greatly simplified and generalised by the recognition of Philbin that a single bath of harmonic oscillators was sufficient to account for dissipation and dispersion [84]. The polarisation field in Huttner and Barnett's model can in fact be written in terms of this oscillator field, removing a degree of complexity while still recovering a generic permittivity ϵ and permeability μ satisfying the Kramers-Kronig relations (I.17). The result is a much more user friendly theory that covers the same range of applications.

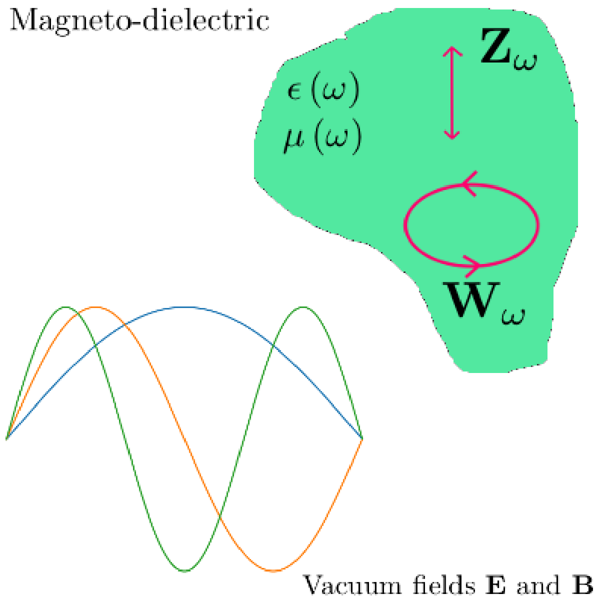

Figure III.1: The quantum vacuum is filled with an infinite electromagnetic energy of all modes. Macroscopic QED models the dispersion and absorption of quantum light by a magneto-dielectric by recovering a complex frequency dependent permativity $\epsilon(\omega)$ and permeability $\mu(\omega)$. The internal degrees of freedom of the magneto-dielectric are modeled by a charge Z_ω and a current W_ω for every frequency ω.

III.2.1 The Action

The Lagrangian density for Macroscopic QED is similar to that of Huttner and Barnett (III.2) except there is no independent polarisation field term,

$$\mathscr{L} = \mathscr{L}_{\text{EM}} + \mathscr{L}_{\text{Osc}} + \mathscr{L}_{\text{Int}} \tag{III.10}$$

where this time, having already been extended to include magnetic media, there are now two reservoirs, one for the electric field, represented by the field amplitude $\mathbf{X}_\omega$, and one for the magnetic field, represented by the field amplitude $\mathbf{Y}_\omega$. The Lagrangian density for the bath of oscillators $\mathscr{L}_{\text{Osc}}$ is now

$$\mathscr{L}_{\text{Osc}} = \frac{1}{2}\int_0^\infty d\omega \left(\dot{\mathbf{X}}_\omega^2 - \omega^2 \mathbf{X}_\omega^2\right) + \mathbf{X}_\omega \to \mathbf{Y}_\omega. \tag{III.11}$$

The interaction Lagrangian is defined similarly to Huttner and Barnett's model

$$\mathscr{L}_{\text{Int}} = \mathbf{E} \cdot \mathbf{P} + \mathbf{B} \cdot \mathbf{M} \tag{III.12}$$

except that the polarisation field is now defined as an integral over all frequencies of the oscillator amplitudes with some position and frequency dependent coupling constant $\alpha(\mathbf{x}, \omega)$, i.e

$$\mathbf{P}(\mathbf{x}, t) = \int_0^\infty d\omega\ \alpha(\mathbf{x}, \omega) \mathbf{X}_\omega(\mathbf{x}, t) \tag{III.13}$$

and where we now have a magnetisation field similarly defined as

$$\mathbf{M}(\mathbf{x}, t) = \int_0^\infty d\omega\ \beta(\mathbf{x}, \omega) \mathbf{Y}_\omega(\mathbf{x}, t). \tag{III.14}$$

The coupling constants are given in terms the dielectric permittivity $\epsilon(\mathbf{x}, \omega)$ and the magnetic permeability $\mu(\mathbf{x}, \omega)$ as

$$\alpha(\mathbf{x}, \omega) = \sqrt{\frac{2\epsilon_0}{\pi} \omega \text{Im} \left[\epsilon(\mathbf{x}, \omega)\right]} \quad \text{and} \quad \beta(\mathbf{x}, \omega) = \sqrt{-\frac{2\kappa_0}{\pi} \omega \text{Im} \left[\kappa(\mathbf{x}, \omega)\right]} \tag{III.15}$$

where $\kappa_0 = \mu_0^{-1}$ and $\kappa(\mathbf{x}, \omega) = \mu(\mathbf{x}, \omega)^{-1}$. The functions $\epsilon(\mathbf{x}, \omega)$ and $\mu(\mathbf{x}, \omega)$ are defined over the *whole* of space. Obviously, in empty space these functions take their free space values, i.e 1. The full permittivity and permeability functions satisfy the Kramer-Kronig relations (I.17). We shall now give a brief overview of how this Lagrangian density, with the coupling constants taking the form (III.15), recover the macroscopic equations of motion for electromagnetism.

III.2.2 The Equations of Motion

Applying the Euler-Lagrange equations (I.35) for each variable leads to the usual macroscopic equations of motion for the electromagnetic fields (I.9) and the usual harmonic oscillator equations of motion for the reservoir fields driven by their coupling to the electric and magnetic fields

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)\mathbf{X}_\omega=\alpha\left(\mathbf{x},\omega\right)\mathbf{E} \tag{III.16}$$

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)\mathbf{Y}_\omega=\beta\left(\mathbf{x},\omega\right)\mathbf{B}. \tag{III.17}$$

The scalar potential ϕ is not a dynamical variable, and is therefore omitted here.

One can proceed analogously to Section I.2 and arrive at expression (I.19) for the example of the electric field. However, it should be remembered that because the oscillators are driven by the electromagnetic field, as can be seen in (III.16) and (III.17), the polarisation and magnetisation are functions of the electric and magnetic fields respectively. The solutions to (III.16) are given as

$$\mathbf{X}_\omega(\mathbf{x},t)=\mathbf{X}^0_\omega(\mathbf{x},t)+\int\limits_{-\infty}^{t}dt'\;G_\omega(t-t')\mathbf{j}_\omega(\mathbf{x},t') \tag{III.18}$$

where, for the sake of causality, the oscillator amplitudes are given in terms of $G_\omega(t-t')$, the retarded harmonic oscillator Green's function which satisfies the equation

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)G_\omega(t-t')=\delta(t-t') \tag{III.19}$$

with the boundary conditions $G_\omega(t-t')=0$ for $t-t'<0$, and the driving current $\mathbf{j}_\omega(\mathbf{x},t)=\alpha\left(\mathbf{x},\omega\right)\mathbf{E}(\mathbf{x},t)$. When taking into account boundary conditions, one must make a choice between keeping the homogeneous solutions in (III.18) or in the homogeneous solutions to the EM field. The choice taken in [84] is the only choice that allows one to quantise the system, namely, the homogeneous solutions to the oscillator field are kept which are given as

$$\mathbf{X}^0_\omega(\mathbf{x},t)=\frac{1}{2\pi}\mathbf{Z}_\omega(\mathbf{x})e^{-i\omega t}+\text{c.c.} \tag{III.20}$$

In Section III.2.3, we will prove that keeping the homogeneous solutions to the electromagnetic field instead makes the quantisation impossible. Similar solutions hold for the magnetic bath of oscillators, where the amplitudes are defined as $\mathbf{W}_\omega(\mathbf{x})$ instead. After having discarded the homogeneous solutions to the electric field, and writing the fields in terms of their Fourier components (I.16), we find expressions for the polarisation (III.13) by multiplying (III.18) by the coupling term $\alpha(\mathbf{x},\omega)$ and integrating over all coupling frequencies. Thus the polarisation in this case is found to be

$$\mathbf{P}(\mathbf{x},t) = \mathbf{P}_0(\mathbf{x},t) + \int_0^\infty d\omega\ \alpha^2(\mathbf{x},\omega) \int_{-\infty}^t dt'\ G_\omega(t-t')\mathbf{E}(\mathbf{x},t). \tag{III.21}$$

where $\mathbf{P}_0(\mathbf{x},t)$ is the polarisation created by the uncoupled parts of the oscillator field (III.20)

$$\mathbf{P}_0(\mathbf{x},t) = \int_0^\infty d\omega\ \alpha(\mathbf{x},\omega)\mathbf{X}^0_\omega(\mathbf{x},t). \tag{III.22}$$

The second term can be written in Fourier space by considering the Fourier components of the retarded oscillator Green's function $G_\omega(t-t')$ which we write as

$$G_{\omega_1}(t-t') = \int_{-\infty}^\infty \frac{d\omega}{2\pi} G_{\omega_1}(\omega) e^{-i\omega(t-t')} \tag{III.23}$$

where

$$G_{\omega_1} = \frac{1}{\omega_1^2 - (\omega - i\eta)^2} \tag{III.24}$$

where, again η is an infinitesimally small positive number serving to shift the pole of the Green's function off the frequency axis (see Appendix A.1). We use the expression for the coupling term (III.15) and the fact that the susceptibility χ and the permittivity ϵ have the same imaginary parts to re-express the Kramers-Kronig relations (I.17) as

$$\chi(\omega) = \frac{1}{\epsilon_0}\int_0^\infty d\omega_1 \frac{\alpha^2(\omega_1)}{\omega_1^2 - (\omega - i\eta)^2} = \frac{2}{\pi}\int_0^\infty d\omega_1 \frac{\omega_1 \mathrm{Im}\left[\chi(\omega_1)\right]}{\omega_1^2 - (\omega - i\eta)^2}. \tag{III.25}$$

We can thus see that the second term on the right hand side of (III.21) takes the same form as the polarisation (I.18). After proceeding in an analogous manner for the magnetisation, one can follow the same derivation to proceed as before until an expression identical to (I.19) except that now the RHS is no longer zero but is given as

$$\nabla \times \left[\mu^{-1}(\mathbf{x},\omega)\nabla \times \mathbf{E}(\mathbf{x},\omega)\right] + \frac{\omega^2}{c^2}\epsilon(\mathbf{x},\omega)\mathbf{E}(\mathbf{x},\omega) = i\mu_0\omega\mathbf{j}(\mathbf{x},\omega) \tag{III.26}$$

where the current $\mathbf{j}$ is defined as

$$\mathbf{j}(\mathbf{x},\omega) = -i\omega\alpha(\mathbf{x},\omega)\mathbf{Z}_\omega(\mathbf{x},\omega) + \nabla \times [\beta(\mathbf{x},\omega)\mathbf{W}_\omega(\mathbf{x},\omega)] \tag{III.27}$$

or, in the time domain

$$\mathbf{j}(\mathbf{x},t) = \mu_0\left(\frac{\partial^2\mathbf{P}_0}{\partial t^2} + \nabla \times \frac{\partial \mathbf{M}_0}{\partial t}\right). \tag{III.28}$$

The interpretation of this current $\mathbf{j}$ is that in Macro QED, the homogeneous solutions of the oscillators $\mathbf{X}^0_\omega$ and $\mathbf{Y}^0_\omega$ represent currents in the medium that *generate* the macroscopic fields. For simple, time-independent systems this current is analogous to the Langevin noise terms used in an alternative description of macroscopic QED [90]. However, the equivalence between these two formalisms is unclear for time-dependent dielectrics. Here, we confine ourselves to using the approach of [84] as its starting point is a well defined Lagrangian with no a priori assumptions on the forms of the currents.

Furthermore, this formulation of the electromagnetic field in terms of a generating current is not unlike the Wheeler-Feynman absorber theory of Section I.4.2. In both cases, the electromagnetic field is written as a shorthand for the effect of the degrees of freedom of matter on other particles. In both cases, if you completely remove the medium, the only thing left is the vacuum fluctuations of the matter degrees of freedom which generate the vacuum fluctuations of the electromagnetic field. In both cases, the EM vacuum fluctuations would disappear if one removed the matter degrees of freedom entirely (however, physically it would make little sense to do so as quantum mechanics requires at least their ground states to exist.)

It is evident that the electric field can now be written in terms of the macroscopic green's function for the magneto-dielectric rather than the vacuum via

$$\mathbf{E}(\mathbf{x},t) = \int d^3\mathbf{x}' \int_0^\infty \frac{d\omega}{2\pi} \left[i\omega\mu_0 \mathbf{G}(\mathbf{x},\mathbf{x}',\omega) \cdot \mathbf{j}(\mathbf{x}',\omega) e^{-i\omega t} + \text{c.c} \right] \tag{III.29}$$

where this Green's function satisfies

$$\nabla \times \left[\mu^{-1}(\mathbf{x},\omega) \nabla \times \mathbf{G}(\mathbf{x},\mathbf{x}',\omega) \right] + \frac{\omega^2}{c^2} \epsilon(\mathbf{x},\omega) \mathbf{G}(\mathbf{x},\mathbf{x}',\omega) = \mathbb{1}_3 \delta(\mathbf{x}-\mathbf{x}'). \tag{III.30}$$

To find an expression for the magnetic field amplitude $\mathbf{B}$, Maxwell's equations (I.9) relate this to the electric field by the expression

$$\mathbf{B}(\mathbf{x},\omega) = \frac{1}{i\omega} \nabla \times \mathbf{E}(\mathbf{x},\omega). \tag{III.31}$$

It is worth making an additional comment about the reservoir degrees of freedom $\mathbf{X}_\omega$. As stated above, this reservoir is a device introduced to account for the energy lost from the electromagnetic field. Yet there is more physical reality to this than merely accounting for lost energy. The reservoir *exactly* mimics the linear response of any medium with local permittivity $\epsilon(\mathbf{x},\omega)$. In any real absorption process radiation excites a time dependent current which could be for example an optical phonon, or an exciton. The reservoir will mimic as much of the physics of this process as is included in the dispersive properties of $\epsilon(\omega)$ (or $\epsilon(\omega,\mathbf{k})$ if non–locality is included [91]). Therefore when we come to calculate excitation rates for coupled matter–field systems in section III.4, the calculated dynamics of the reservoir must be read as a proxy for the electromagnetic field exciting a real quasiparticle within the medium. We expect the approximations inherent in our use of the reservoir to describe the medium to be the same as those involved in assuming that the medium can be characterized by a linear response $\epsilon(\mathbf{x},\omega)$.

III.2.3 No-Go Theorem for Alternative Expansion in Macro QED

In Section III.2.2, we mentioned that only the choice of homogeneous solution made in [84] allows for the quantisation of the electromagnetic field within dispersive, dissipative magneto-dielectrics. There are two alternative choices: either we keep the homogeneous solutions to the electromagnetic, or we keep both set of homogeneous solutions. Here we will consider the former and prove that if we discard the homogeneous solutions of the oscillator bath and keep those of the electromagnetic field, the resulting system cannot be canonically quantised. A similar argument holds for the latter case but, to our knowledge, has not yet been explicitely proven.

The physical picture considered now is that the full dynamics of the interaction between the electromagnetic field and the medium is generated by uncoupled modes of the electromagnetic field, i.e plane waves coming in from infinity from all directions. These modes polarise the medium that then creates its own field by reaction and so forth. We will not discuss the case where neither of the homogeneous solutions are discarded.

For simplicity, we consider the much simpler, $1+1$ dimensional scalar field case but the results also hold for the usual $3+1$ dimensional vector field case. The analogous equation of motion for the field is

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi\left(x,t\right)=\dot{P}\left(x,t\right) \tag{III.32}$$

where the scalar polarisation is given, analogously to (III.13), as

$$P=\int_0^\infty d\omega\ \alpha(x,\omega)X_\omega \tag{III.33}$$

and $\alpha(x,\omega)=\sqrt{2\omega\epsilon_0\epsilon_I(x,\omega)/\pi}$. The equation of motion for each oscillator of frequency ω is

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)X_\omega\left(x,t\right)=\alpha(x,\omega)\dot{\phi}\left(x,t\right). \tag{III.34}$$

As previously described, the usual approach of macroscopic QED is to solve the equations of motion for both fields but discard the homogeneous solutions to (III.32), keeping only those of (III.34). In other words, we write

$$\phi\left(x,t\right)=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dt'\;g(x,x',t,t')\dot{P}(x',t') \tag{III.35}$$

where the kernel of the integral is the free space green's function and

$$X_{\omega}\left(x,t\right)=X_{\omega}^{H}\left(x,t\right)+\int_{-\infty}^{\infty}dt'\;G_{\omega}(t,t')\alpha(x,\omega)\dot{\phi}\left(x,t'\right) \tag{III.36}$$

where X_{ω}^{H} is the solution to (III.34) with the right hand side set to zero and the kernel $G_{\omega}(t,t')$ is the green's function of the harmonic oscillator of frequency ω.

The justification for having to make this choice is that if both homogeneous solutions are kept, for there to be finite fields inside any material, the boundaries at infinity lead to infinite amplitudes for the scalar field due to the omnipresence of a small absorption over all space, i.e the vacuum is considered to be a limit of vanishing absorption but not absorption-less per se.

It is again possible, through Fourier transforming equations (III.35) and (III.36) into the frequency domain, to express the equations of motion of the scalar field (III.32) as the macroscopic wave equation for a scalar field in a inhomogeneous absorbing medium subject to a current, i.e

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\omega^2}{c^2}\epsilon(x,\omega)\right]\phi(x,\omega)=-i\omega j_0(x,\omega). \tag{III.37}$$

where $j_0(x,\omega)$ is a function of the homogeneous solutions to the oscillator equation (III.34). The interpretation is of course as before, in this model, the homogeneous solutions of the oscillator bath inside the medium generate the field over all space.

The choice of which homogeneous solution to retain may seem to be done rather arbitrarily. Here, we explore the alternative choice and assess whether it leads to anything physically meaningful or useful. Therefore, in this case, we keep the homogeneous solutions to (III.32) and the analogous expressions for the solutions (III.35) and (III.36) are

$$\phi\left(x,t\right)=\phi_{H}\left(x,t\right)+\int\limits_{-\infty}^{\infty}dx'\int\limits_{-\infty}^{\infty}dt'\;g(x,x',t,t')\dot{P}(x',t') \tag{III.38}$$

and

$$X_{\omega}\left(x,t\right)=\alpha(x,\omega)\int\limits_{-\infty}^{\infty}dt'\;G_{\omega}(t,t')\dot{\phi}\left(x,t'\right). \tag{III.39}$$

Let us now consider the equation of motion of the oscillator field given by (III.34). Substituting the solutions to the scalar field (III.38) into the right hand side gives

$$\left(\frac{\partial^{2}}{\partial t^{2}}+\omega^{2}\right)X_{\omega}\left(x,t\right)=\alpha(x,\omega)\partial_{t}\left\{\phi_{H}\left(x,t\right)+\int\limits_{-\infty}^{\infty}dx'\int\limits_{-\infty}^{\infty}dt'\;g(x,x',t,t')\partial_{t'}P(x',t')\right\}. \tag{III.40}$$

Let us now Fourier transform this equation into the frequency domain,

$$X_{\omega}\left(x,\omega_{0}\right)=-\frac{i\omega_{0}\alpha(x,\omega)}{\omega^{2}-(\omega_{0}+i\eta)^{2}}\left\{\phi_{H}\left(x,\omega_{0}\right)-i\omega_{0}\int\limits_{-\infty}^{\infty}dx'\;g(x,x',\omega_{0})P(x',\omega_{0})\right\} \tag{III.41}$$

where η is an infinitesimal positive number introduced to deal with the pole at $\omega=\omega_0$. Multiplying this equation by $\alpha(x,\omega)$ and integrating over all ω, we arrive at an equation for the polarisation

$$P\left(x,\omega_{0}\right)+\frac{\omega_{0}^{2}}{c^{2}}\chi(x,\omega_{0})\int\limits_{-\infty}^{\infty}dx'\;g(x,x',\omega_{0})P(x',\omega_{0})=-i\frac{\omega_{0}}{c^{2}}\chi(x,\omega_{0})\phi_{H}\left(x,\omega_{0}\right). \tag{III.42}$$

The right hand side of (III.42) is the homogeneous solution to the vacuum scalar field equations (III.32) and as such can be represented as a sum of monochromatic plane waves. Here, we use $\phi_H(x,\omega_0)=\phi_0\exp(i\omega_0 x/c)$ and use the expression for the vacuum Green's function $g(x,x',\omega_0)=ic\exp\left(i\omega_0|x-x'|/c\right)/2\omega_0$ leading to

$$P\left(x,\omega_{0}\right)+i\frac{\omega_{0}}{2c}\chi(x,\omega_{0})\int\limits_{-\infty}^{\infty}dx'\;e^{i\frac{\omega_{0}}{c}|x-x'|}P(x',\omega_{0})=-i\frac{\omega_{0}}{c^{2}}\chi(x,\omega_{0})\phi_{0}e^{i\frac{\omega_{0}}{c}x}. \tag{III.43}$$

Let us consider a block of material whose spatial variation is given in figure III.2. In this case, if $x < 0$, then, since in the vacuum $\chi(\omega) = 0$, equation (III.43) becomes $P(x, \omega_0) = 0$, as one would expect. As such, equation (III.43) can be written as

$$P(x, \omega_0) + i\frac{\omega_0}{2c}\chi(\omega_0)\int_0^\infty dx'\ e^{i\frac{\omega_0}{c}|x-x'|}P(x', \omega_0) = -i\frac{\omega_0}{c^2}\chi(\omega_0)\phi_0 e^{i\frac{\omega_0}{c}x} \tag{III.44}$$

where $x \geq 0$. Consider a trial solution of the form $P(x, \omega_0) = P_0 \exp[i\omega_0 f(\omega_0)x/c]$. Substituting this into (III.44), leads to

$$P_0 e^{i\frac{\omega_0}{c}f(\omega_0)x}\left[1 + \frac{\chi(\omega_0)}{f^2(\omega_0) - 1}\right] - \frac{1}{2}\frac{\chi(\omega_0)}{f(\omega_0) - 1}P_0 e^{i\frac{\omega_0}{c}x} = -i\frac{\omega_0}{c^2}\chi(\omega_0)\phi_0 e^{i\frac{\omega_0}{c}x} \tag{III.45}$$

where we have used the result

$$\begin{aligned}
\int_0^\infty dx'\ e^{i\frac{\omega_0}{c}|x-x'|}e^{i\frac{\omega_0}{c}f(\omega_0)x'} &= e^{i\frac{\omega_0}{c}x}\int_0^x dx'\ e^{i\frac{\omega_0}{c}[f(\omega_0)-1]x'} + e^{-i\frac{\omega_0}{c}x}\int_x^\infty dx'\ e^{i\frac{\omega_0}{c}[f(\omega_0)+1]x'} \\
&= e^{i\frac{\omega_0}{c}x}\left[\frac{e^{i\frac{\omega_0}{c}[f(\omega_0)-1]x'}}{i\frac{\omega_0}{c}[f(\omega_0)-1]}\right]_0^x + e^{-i\frac{\omega_0}{c}x}\left[\frac{e^{i\frac{\omega_0}{c}[f(\omega_0)+1]x'}}{i\frac{\omega_0}{c}[f(\omega_0)+1]}\right]_x^\infty \\
&= \frac{e^{i\frac{\omega_0}{c}f(\omega_0)x} - e^{i\frac{\omega_0}{c}x}}{i\frac{\omega_0}{c}[f(\omega_0)-1]} - \frac{e^{i\frac{\omega_0}{c}f(\omega_0)x}}{i\frac{\omega_0}{c}[f(\omega_0)+1]}
\end{aligned} \tag{III.46}$$

where we have $P(x, \omega_0) \to 0$ as $x \to +\infty$. If we set $P_0 = i2\omega_0[f(\omega_0) - 1]\phi_0/c^2$, then (III.45) leads to

$$f(\omega_0) = \pm\sqrt{1 - \chi(\omega_0)} \tag{III.47}$$

that in turn leads to

$$P(x, \omega_0) = i\frac{2\omega_0}{c^2}\left[\sqrt{1 - \chi(\omega_0)} - 1\right]\phi_0 e^{i\frac{\omega_0}{c}\sqrt{1-\chi(\omega_0)}x}. \tag{III.48}$$

The polarisation is plotted in figure III.2. Again, using the results (III.46), we rewrite the expression for the oscillator field amplitude (III.41) as

$$X_\omega(x, \omega_0) = -\frac{\alpha(x, \omega)}{\omega^2 - (\omega_0 + i\omega 0^+)^2} \cdot \frac{i\omega_0}{\sqrt{1 - \chi(\omega_0)} + 1}\phi_0 e^{i\frac{\omega_0}{c}\sqrt{1-\chi(\omega_0)}\ x}. \tag{III.49}$$

This amplitude contains no terms that are independent of the absorption of the material and as such, far into the material, even the quantum uncertainty is absorbed. This makes quantising the theory infeasible as the commutation relations cannot be satisfied.

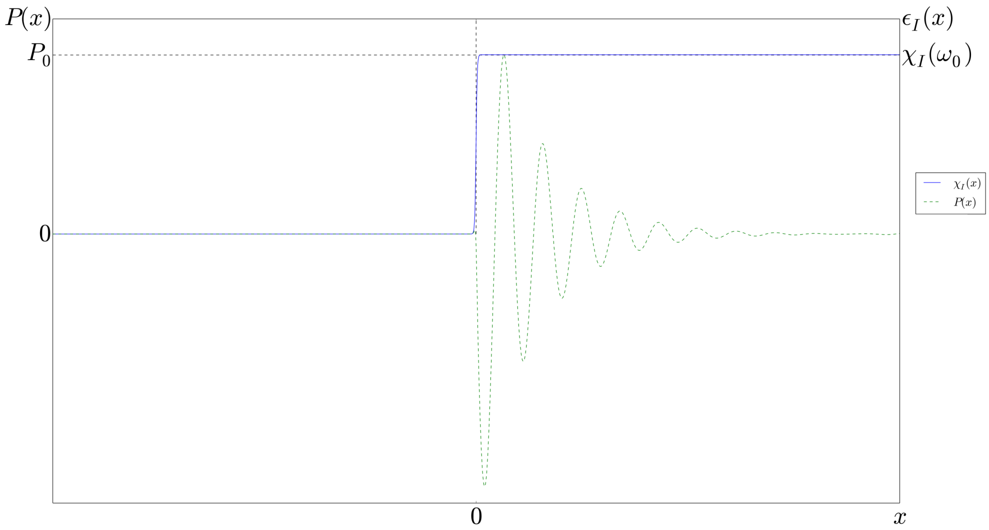

Figure III.2: Polarisation $P(x, \omega_0)$ inside a half space dielectric of constant permittivity $\epsilon_I(\omega)$.

Now, it is also possible to derive an equivalent macroscopic equation of motion to that of (III.37). To achieve this, consider the equation for the frequency components of the scalar field

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\right)\phi\left(x,\omega\right) = -i\omega P\left(x,\omega\right) \tag{III.50}$$

where from (III.34), by multiplying by $\alpha(x, \omega)$ and integrating over ω, we write the Fourier components of the polarisation as

$$P\left(x,\omega\right) = -i\frac{\omega}{c^2}\chi(x,\omega)\phi\left(x,\omega\right). \tag{III.51}$$

Substituting (III.51) into (III.50) gives

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\epsilon(x,\omega)\right]\phi\left(x,\omega\right) = 0. \tag{III.52}$$

The solutions to these, like the oscillator field, can be seen to decay into the medium. In a sense, these are the modes of light that we are looking for, but expressed in this manner makes the quantisation procedure impossible unless we artificially add increasing terms to ensure conservation of energy. This would essentially make having introduced the oscillator field pointless. Furthermore, such additional terms have already been used to quantise such systems and are known as Langevin-noise terms (see for example [90] and citations therein). In fact, the addition of such terms has been shown to be equivalent to this bath of oscillators model where we keep the solutions to the oscillator field instead. Thus, it seems that the choice of source in the approach of [84] seems to be the only one that permits a rigorous quantisation.

III.2.4 Second Quantisation

Continuing on from the formalism of Section III.2.2, one can construct the canonical momenta for the fields in the usual manner using (I.35) where the momentum of the potential field becomes $\Pi_{\mathbf{A}} = -\epsilon_0 \mathbf{E} - \mathbf{P}$ and the momentum of the each oscillator of frequency ω is identical to that given by (I.27). The Hamiltonian of the system can then be constructed in the usual manner via (I.37). Analogously to Hopfield's model and later that of Huttner and Barnett, the Hamiltonian can be diagonalised by the polariton quasi-particles representing a *mixture* of excitations of the oscillator fields and the electromagnetic fields. However, note that in this model, the polariton creation and annihilation operators carry an extra subscripts, e or m, representing the electric and magnetic field respectively. In other words, there is a set of creation and annihilation operators for each of the two baths. The diagonalised Hamiltonian operator is

$$\hat{H} = \sum_{\lambda=\mathrm{e,m}} \int d^3\mathbf{x} \int_0^\infty d\omega\ \hbar\omega\ \hat{\mathbf{C}}^\dagger_\lambda\left(\mathbf{x},\omega\right) \cdot \hat{\mathbf{C}}_\lambda\left(\mathbf{x},\omega\right) \tag{III.53}$$

where the polariton creation and annihilation operators satisfy

$$\left[\hat{\mathbf{C}}_\lambda\left(\mathbf{x},\omega\right),\hat{\mathbf{C}}^\dagger_{\lambda'}\left(\mathbf{x}',\omega'\right)\right] = \hat{\mathbf{C}}_\lambda\left(\mathbf{x},\omega\right)\otimes\hat{\mathbf{C}}^\dagger_{\lambda'}\left(\mathbf{x}',\omega'\right) - \left(\hat{\mathbf{C}}^\dagger_{\lambda'}\left(\mathbf{x}',\omega'\right)\otimes\hat{\mathbf{C}}_\lambda\left(\mathbf{x},\omega\right)\right)^T$$
$$= \mathbb{1}_3\delta_{\lambda\lambda'}\delta(\mathbf{x}-\mathbf{x}')\delta(\omega-\omega'). \tag{III.54}$$

The operators of electromagnetic field, $\hat{\mathbf{A}}$, $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ are thus expanded in the following form

$$\hat{\mathbf{O}}(\mathbf{x},t) = \sum_{\lambda=\text{e,m}}\int d^3\mathbf{x}'\int_0^\infty d\omega\left[\mathbf{f}^\lambda_O(\mathbf{x},\mathbf{x}',\omega)\cdot\hat{\mathbf{C}}_\lambda(\mathbf{x}',\omega)e^{-i\omega t}+\text{h.c}\right] \tag{III.55}$$

while there is an additional variable for the oscillator operators $\hat{\mathbf{X}}_\omega$ and $\hat{\mathbf{Y}}_\omega$

$$\hat{\mathbf{O}}_\omega(\mathbf{x},t) = \sum_{\lambda=\text{e,m}}\int d^3\mathbf{x}'\int_0^\infty d\omega'\left[\mathbf{f}^\lambda_O(\mathbf{x},\mathbf{x}',\omega,\omega')\cdot\hat{\mathbf{C}}_\lambda(\mathbf{x}',\omega')e^{-i\omega' t}+\text{h.c}\right]. \tag{III.56}$$

Here we list the expansion coefficients derived in [84] for reference throughout the following sections

$$\begin{aligned}
&\mathbf{f}_\mathbf{B}\left(\mathbf{x},\mathbf{x_1},\omega_1\right) = -i\mu_0\sqrt{\frac{\hbar}{2\omega_1}}\omega_1\alpha\left(\mathbf{x_1},\omega_1\right)\boldsymbol{\nabla}\times\mathbf{G}\left(\mathbf{x},\mathbf{x_1},\omega_1\right)\\
&\mathbf{f}_\mathbf{X}\left(\mathbf{x},\mathbf{x_2},\omega,\omega_2\right) = \sqrt{\frac{\hbar}{2\omega_2}}\left[\frac{\mu_0\omega_2^2\alpha\left(\mathbf{x},\omega\right)\alpha\left(\mathbf{x_2},\omega_2\right)}{\omega^2-\left(\omega_2+i\omega0^+\right)^2}\mathbf{G}\left(\mathbf{x},\mathbf{x_2},\omega_2\right)+\delta^{(3)}\left(\mathbf{x}-\mathbf{x_2}\right)\delta\left(\omega-\omega_2\right)\mathbb{1}_3\right]\\
&\mathbf{f}_\mathbf{E}\left(\mathbf{x},\mathbf{x_1},\omega_1\right) = \mu_0\sqrt{\frac{\hbar}{2\omega_1}}\omega_1^2\alpha\left(\mathbf{x_1},\omega_1\right)\mathbf{G}\left(\mathbf{x},\mathbf{x_1},\omega_1\right)\\
&\mathbf{f}_\mathbf{A}\left(\mathbf{x},\mathbf{x_1},\omega_1\right) = -\frac{i}{\omega}\left[\mathbf{f}_\mathbf{E}\left(\mathbf{x},\mathbf{x_1},\omega_1\right)\right]_T\\
&\mathbf{f}_{\mathbf{\Pi_A}}\left(\mathbf{x},\mathbf{x_1},\omega_1\right) = -\epsilon_0\epsilon(\mathbf{x},\omega_1)\mathbf{f}_\mathbf{E}\left(\mathbf{x},\mathbf{x_1},\omega_1\right)-\sqrt{\frac{\hbar}{2\omega_1}}\alpha(\mathbf{x},\omega_1)\delta^{(3)}(\mathbf{x}-\mathbf{x_1})\\
&\mathbf{f}_{\mathbf{\Pi_X}}\left(\mathbf{x},\mathbf{x_1},\omega,\omega_1\right) = -i\omega_1\mathbf{f}_\mathbf{X}\left(\mathbf{x},\mathbf{x_1},\omega,\omega_1\right).
\end{aligned} \tag{III.57}$$

It is also possible to perform a similar diagonalisation procedure in $(\mathbf{k},\omega)$-space analogously to Huttner and Barnett's model, in which case in can be similarly seen that the quasi-particles of this system again do not follow a strict dispersion curve but instead occupy the whole $(\mathbf{k},\omega)$-space. In fact, this is the approach taken in the case of moving media discussed in section III.3.2. The electric field operator can then be expressed in terms the operator form of the current (III.27) as

$$\hat{\mathbf{E}}(\mathbf{x},t) = \mu_0 \int d\mathbf{x}_1 \int_0^\infty \frac{d\omega_1}{2\pi} \left[i\omega_1 \mathbf{G}(\mathbf{x},\mathbf{x}_1,\omega_1) \cdot \hat{\mathbf{j}}(\mathbf{x}_1,\omega_1) e^{-i\omega_1 t} + \text{h.c} \right] \tag{III.58}$$

and the magnetic field operator can be found using the operator form of the relation (III.27). The current operator is given by

$$\mathbf{j}(\mathbf{x},\omega) = \sqrt{\frac{\hbar}{2\omega}} \left\{ -2\pi i\omega\alpha(\mathbf{x},\omega)\, \hat{\mathbf{C}}_{\mathsf{e}}(\mathbf{x},\omega) + 2\pi\nabla \times \left[\beta(\mathbf{x},\omega) \hat{\mathbf{C}}_{\mathsf{m}}(\mathbf{x},\omega) \right] \right\}. \tag{III.59}$$

One of the biggest strengths of macroscopic QED can be seen in the simple expression (III.58). One may consider large scale, i.e *macroscopic*, magneto-dielectric entities, subject to quantum mechanical fluctuations of the fields. The presence of the Green's function allows calculations to be carried out using relatively complex geometries, and the current $\hat{\mathbf{j}}$ plays the role of the quantum mechanical fluctuations that generate the fields.

This model has the advantage of being a fully quantised canonical theory subject to the usual tool set of quantum field theory while simulating the absorption and dissipation of electromagnetic energy in magneto-dielectrics satisfying the constitutive equations $\mathbf{D}(\omega) = \epsilon(\omega)\mathbf{E}(\omega)$ and $\mathbf{H}(\omega) = \mu^{-1}(\omega)\mathbf{B}(\omega)$. All this while still being comparatively simple. We will now consider some of the extensions that have been applied to the theory which will include those for moving media on which our future sections will depend.

III.3 Arbitrary Motion

III.3.1 Extending the applicability of macro QED

Macroscopic QED has the advantage that, to some extent, the bath of oscillators represents the minimal degrees of freedom of matter required to simulate the absorption and dissipation of electromagnetic energy in magneto-dielectrics satisfying the constitutive equations $\mathbf{D}(\omega) = \epsilon(\omega)\mathbf{E}(\omega)$ and $\mathbf{H}(\omega) = \mu^{-1}(\omega)\mathbf{B}(\omega)$. If one wishes to take into account spatial dispersion, non-linear effects or the eventual re-emission of absorbed energy through blackbody radiation, significant modifications to the theory would be required. However, if one wished to be able to recover constitutive equations of the form $\mathbf{D}(\omega) = \boldsymbol{\epsilon}(\omega)\cdot\mathbf{E}(\omega) + \boldsymbol{\chi}_{\mathsf{EB}}(\omega)\cdot\mathbf{B}(\omega)$ and $\mathbf{H}(\omega) = \boldsymbol{\mu}(\omega)\cdot\mathbf{B}(\omega) + \boldsymbol{\chi}_{\mathsf{BE}}(\omega)\cdot\mathbf{E}$ which include, but are not limited to, chiral media and moving media, then modifying the coupling coefficients, and, in the case of moving media, Lorentz boosting the reservoir, is sufficient [85]. Note that the permittivity ϵ has now become a tensor $\boldsymbol{\epsilon}(\omega) = \epsilon_0\left[\mathbb{1}_3 + \boldsymbol{\chi}_{\mathsf{EE}}(\omega)\right]$, as has the permeability μ. In this description, the polarisation term appearing in the Lagrangian coupling term (III.12) is now described by the more general expression

$$\mathbf{P} = \int_0^\infty d\omega\left[\left(\boldsymbol{\alpha}_{\mathsf{EE}} + \boldsymbol{\beta}_{\mathsf{EE}}\frac{\partial}{\partial t}\right)\cdot\mathbf{X}_\omega + \left(\boldsymbol{\alpha}_{\mathsf{EB}} + \boldsymbol{\beta}_{\mathsf{EB}}\frac{\partial}{\partial t}\right)\cdot\mathbf{Y}_\omega\right] \tag{III.60}$$

with an analogous expression for the magnetisation. Here the $\boldsymbol{\beta}$ tensors represent a different form of coupling and do not represent the same magnetic coupling tensors of [84] described in Section III.2. In order to differentiate between mangetic and electric couplings, the subscripts EE, EB, BE and BB were introduced. The scalar coupling terms of [84] are assimilated into two of the tensors, i.e. $\alpha(\mathbf{x},\omega) \to \boldsymbol{\alpha}_{\mathsf{EE}}(\mathbf{x},\omega)$ and $\beta(\mathbf{x},\omega) \to \boldsymbol{\alpha}_{\mathsf{BB}}(\mathbf{x},\omega)$ in (III.13) and (III.14) respectively where the tensor susceptibilities are related to the coupling tensors in an analogous but more general fashion than (III.15). The restrictions placed upon the coupling tensors for various physical scenarios are also investigated in [85].

III.3.2 Constant Motion

In the case of media moving with a constant velocity $\mathbf{v}$, the form of the coupling tensors can be derived explicitly as in [85] by considering how the fields transform when moving from one reference frame to another. Starting from (III.10), if we consider the reference frame to be in constant relative motion with respect to the bath of oscillators, the free field density is of course unchanged, but the coupling term is given by

$$\mathscr{L}_{\mathsf{Int}} = \mathbf{E}' \cdot \mathbf{P}' + \mathbf{B}' \cdot \mathbf{M}' \qquad \text{(III.61)}$$

where the prime denotes a Lorentz transformation of the quantities. The primed field amplitudes are related to the unprimed coordinates by (I.7) and (I.8). In order to proceed, we consider the amplitudes of the oscillators themselves to be unchanged under a Lorentz transformation. Rewriting (III.61) in terms of the unprimed amplitudes leads to expressions for the primed coupling tensors in terms of the unprimed tensors

$$\boldsymbol{\alpha}'_{\mathsf{EE}}(\omega) = \boldsymbol{\Lambda} \cdot \boldsymbol{\alpha}_{\mathsf{EE}}(\omega) \qquad \text{(III.62)}$$

$$\boldsymbol{\alpha}'_{\mathsf{BB}}(\omega) = \boldsymbol{\Lambda} \cdot \boldsymbol{\alpha}_{\mathsf{BB}}(\omega) \qquad \text{(III.63)}$$

$$\boldsymbol{\alpha}'_{\mathsf{EB}}(\omega) = \frac{\gamma \mathbf{v}}{c^2} \times \boldsymbol{\alpha}_{\mathsf{BB}}(\omega) \qquad \text{(III.64)}$$

$$\boldsymbol{\alpha}'_{\mathsf{BE}}(\omega) = -\gamma \mathbf{v} \times \boldsymbol{\alpha}_{\mathsf{EE}}(\omega) \qquad \text{(III.65)}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(1, \gamma, \gamma)$. Here, we consider the simpler case where the $\boldsymbol{\beta}$ tensors are all taken to be zero. Using the transformation of derivatives (I.5) the Lagrangian density for the reservoir in the lab frame is given by

$$\mathscr{L}_{\mathsf{Osc}} = \frac{1}{2} \int_0^\infty d\omega \left\{ \gamma^2 \left[\frac{\partial \mathbf{X}_\omega}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{X}_\omega \right]^2 - \omega^2 \mathbf{X}_\omega^2 \right\} + \mathbf{X}_\omega \to \mathbf{Y}_\omega. \qquad \text{(III.66)}$$

In this case, the new modified polarisation and magnetisation are given by

$$\mathbf{P}(\mathbf{x}, t) = \int_0^\infty d\omega \left[\boldsymbol{\alpha}_{\mathsf{EE}}(\omega, \mathbf{x}, t) \cdot \mathbf{X}_\omega(\mathbf{x}, t) + \boldsymbol{\alpha}_{\mathsf{EB}}(\omega, \mathbf{x}, t) \cdot \mathbf{Y}_\omega(\mathbf{x}, t) \right] \qquad \text{(III.67)}$$

and

$$\mathbf{M}(\mathbf{x}, t) = \int_0^\infty d\omega \left[\boldsymbol{\alpha}_{\mathsf{BB}}(\omega, \mathbf{x}, t) \cdot \mathbf{Y}_\omega(\mathbf{x}, t) + \boldsymbol{\alpha}_{\mathsf{BE}}(\omega, \mathbf{x}, t) \cdot \mathbf{X}_\omega(\mathbf{x}, t) \right] \qquad \text{(III.68)}$$

where the coupling terms are now dependent on position *and* time due to the Lorentz transformation of the spatial dependencies of the co-moving coupling tensors using (I.2). This modification is in fact quite important. There has been, in the past, some confusion regarding the resemblance of the constitutive equations for moving media and bi-anisotropic media [12] (i.e. the motion creates the appearance of an effective medium with magnetic, electric and magneto-electric coupling). The analogy between the two suggests at a first glance that it should be possible to draw energy from a material for free. However, the extensions to macroscopic QED suggest a resolution to this paradox by demonstrating a clear difference between the two which we will now briefly illustrate.

Bi-anisotropic Media and Moving Dielectrics

To illustrate this notion, let us ignore magnetic effects by setting $\mu \to 1$ and removing the second bath of oscillators. They are superfluous for this present consideration. Now consider an isotropic, homogeneous dielectric filling the entirety of space and moving at a constant velocity $\mathbf{v}$.

In our model, only two coupling terms are non-trivial, namely $\boldsymbol{\alpha}_{\mathsf{EE}}$ and $\boldsymbol{\alpha}_{\mathsf{BE}}$, both of which can be reduced to scalar form in this case and no longer depend on position or time. Only one bath of oscillators is retained since we are ignoring the magnetic part. Without disrupting our present argument, we can also ignore the second of these remaining coupling terms, although this would need to be included for any accurate answer to be obtained. The present argument holds with or without this term. These things considered, for moving media, the analogous equations to (III.16) is then

$$\left[\gamma^2 \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)^2 + \omega^2\right] \mathbf{X}_\omega\left(\mathbf{x}, t\right) = \alpha_{\mathsf{EE}}(\omega)\mathbf{E}\left(\mathbf{x}, t\right) \tag{III.69}$$

which is just the Lorentz boosted driven harmonic oscillator equation of motion.

Expanding the oscillator field into its spatial and temporal Fourier components via

$$\mathbf{X}_{\omega_1}(\mathbf{x},t)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\mathbf{X}_{\omega_1}(\mathbf{k},\omega)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{III.70}$$

allows us to find the solutions to (III.69) in a similar manner to (III.18) given by

$$\mathbf{X}_{\omega_1}(\mathbf{x},t)=\mathbf{X}^0_{\omega_1}(\mathbf{x},t)+\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{\omega_1^2-(\omega_-+i\eta)^2}\alpha_{\mathsf{EE}}(\omega)\mathbf{E}(\mathbf{k},\omega)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{III.71}$$

where $\omega_-=\gamma\,(\omega-\mathbf{v}\cdot\mathbf{k})$. To compare the two, note that (III.18) could have equally been expressed as

$$\mathbf{X}_{\omega_1}(\mathbf{x},t)=\mathbf{X}^0_{\omega_1}(\mathbf{x},t)+\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{\omega_1^2-(\omega+i\eta)^2}\alpha_{\mathsf{EE}}(\omega)\mathbf{E}(\mathbf{k},\omega)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{III.72}$$

The difference to note here is the subscript "$-$" denoting a Lorentz transformation of the frequency $\omega_-=\gamma\,(\omega-\mathbf{v}\cdot\mathbf{k})$. Therefore the effect of the Lorentz transformation of the derivatives in (III.69) is to introduce a Doppler shift into the frequency of (III.71). As before, multiplying by the appropriate coupling term and integrating over all ω_1, recovers the expression for the polarisation,

$$\mathbf{P}(\mathbf{x},t)=\mathbf{P}_0(\mathbf{x},t)+\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\chi(\omega_-)\,\mathbf{E}(\mathbf{k},\omega)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{III.73}$$

where the second term on the right hand side, goes on to give the permittivity term of the full macroscopic equation. The difference here is that now this susceptibility is the Doppler shifted susceptibility $\chi(\omega_-)$ defined as

$$\chi(\omega_-)=\int\limits_0^{\infty}d\omega_1\frac{\alpha^2_{\mathsf{EE}}(\omega)}{\omega_1^2-(\omega_-+i\eta)^2}. \tag{III.74}$$

Had the degrees of freedom of the medium not been taken into account, this shift in the material response would not have been obvious. This led to a paradox whereby it was supposed that a bi-anisotropic medium might be able to derive an arbitrary amount of energy from nothing by mimicking a moving dielectric. This extension to [84] suggests that the two are not in fact equivalent.

We will now consider a calculation recently performed in [44] using these extensions to macroscopic QED to establish the excitation rate of a two-level system outside a dielectric half space which closely resembles the cases discussed in Chapter II.

Excitation Rate of a Two-Level System

The model of [84] is the natural amalgamation of the systems considered in Sections II.1 and II.3 with the additional inclusion of dissipation and dispersion of light within the medium. The relative motion and the relative distance between the two-level system and the half-space dielectric are held fixed in such a manner that the transfer of energy between the two entities manifests itself entirely as an excitation of the two level system.

In order to establish the excitation rate of the two-level particle, the coupling between it and the electromagnetic field is treated as a perturbation to the case containing only the electromagnetic field and the reservoirs. The half space dielectric coupled to the electromagnetic field is quantised in a similar manner to the original work on macroscopic QED of [84]. However, in [44] the half space dielectric is set in motion relative to the frame of reference. It is also taken to be spatially homogeneous in the direction of motion with an arbitrary spatial dependence in the other directions.

The reasoning behind considering the material to be spatially homogeneous in the direction of motion, is that there are two effects that we wish to distinguish in doing so. Before the inclusion of the matter degrees of freedom in the Lagrangian of Huttner and Barnett's original model, it was assumed that a spatially homogeneous medium in motion would exhibit the same effects as when at rest as the electromagnetic field would not distinguish between the two due to spatial symmetries. The addition of spatial inhomogeneities would of course create its own modifications to the electromagnetic field. In order to consider this effectively, one must consider the *simplest* case of spatial homogeneity. It is only necessary for this homogeneity to be in the direction of motion, so the other directions are left as they are in the general case of [84], i.e. a yet undefined spatial dependence. In the next section, we will consider the case where there may exist homogeneities in the direction of motion and what effect this has on the quantum vacuum.

The result of this is that coupling coefficients, which are related to the permittivity and the permeability via (III.15) are independent of position in the direction of motion and only present a spatial dependence perpendicular to the direction of motion. The spatial dependence of the coupling terms therefore exhibit no Lorentz boost from the motion, as can be seen from the transformation of the coordinates (I.2). Taking these factors into account, the Lagrangian density is as described at the start of this section except that our polarisation (III.67) can now written in the simpler form

$$\mathbf{P}\left(\mathbf{x},t\right)=\int_{0}^{\infty}d\omega\left[\boldsymbol{\alpha}_{\mathsf{EE}}\left(\omega,\mathbf{x}_{\perp}\right)\cdot\mathbf{X}_{\omega}(\mathbf{x},t)+\boldsymbol{\alpha}_{\mathsf{EB}}\left(\omega,\mathbf{x}_{\perp}\right)\cdot\mathbf{Y}_{\omega}(\mathbf{x},t)\right] \tag{III.75}$$

where $\mathbf{x}_{\perp}$ is the part of the position vector in the plane perpendicular to the motion of the dielectric. Similarly for the magnetisation, we have

$$\mathbf{M}\left(\mathbf{x},t\right)=\int_{0}^{\infty}d\omega\left[\boldsymbol{\alpha}_{\mathsf{BB}}\left(\omega,\mathbf{x}_{\perp}\right)\cdot\mathbf{Y}_{\omega}(\mathbf{x},t)+\boldsymbol{\alpha}_{\mathsf{BE}}\left(\omega,\mathbf{x}_{\perp}\right)\cdot\mathbf{X}_{\omega}(\mathbf{x},t)\right]. \tag{III.76}$$

The equations of motion for the reservoirs are of course very similar to (III.69),

$$\left[\gamma^{2}\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)^{2}+\omega^{2}\right]\mathbf{X}_{\omega}\left(\mathbf{x},t\right)=\boldsymbol{\alpha}_{\mathsf{EE}}(\omega,\mathbf{x}_{\perp})\cdot\mathbf{E}\left(\mathbf{x},t\right)+\boldsymbol{\alpha}_{\mathsf{BE}}^{T}(\omega,\mathbf{X}_{\perp})\cdot\mathbf{B}\left(\mathbf{x},t\right) \tag{III.77}$$

and

$$\left[\gamma^{2}\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)^{2}+\omega^{2}\right]\mathbf{Y}_{\omega}\left(\mathbf{x},t\right)=\boldsymbol{\alpha}_{\mathsf{BB}}(\omega,\mathbf{x}_{\perp})\cdot\mathbf{B}\left(\mathbf{x},t\right)+\boldsymbol{\alpha}_{\mathsf{EB}}^{T}(\omega,\mathbf{X}_{\perp})\cdot\mathbf{E}\left(\mathbf{x},t\right) \tag{III.78}$$

Again, analogously to the stationary case of [84], substituting into (III.75) and

(III.76) the solutions to these equations of motion for reservoirs, and then in term, substituting those into the equations of motion of the fields, allows us to identify the more general effective susceptibilities whose frequencies are once again, Doppler shifted, namely

$$\chi_{\mathsf{EE}}\left(\omega_{-},\mathbf{x}_{\perp}\right)=\int_{0}^{\infty}d\omega_{1}\frac{\boldsymbol{\alpha}_{\mathsf{EE}}(\omega,\mathbf{x}_{\perp})\cdot\boldsymbol{\alpha}_{\mathsf{EE}}^{T}(\omega,\mathbf{x}_{\perp})+\boldsymbol{\alpha}_{\mathsf{BB}}(\omega,\mathbf{x}_{\perp})\cdot\boldsymbol{\alpha}_{\mathsf{BB}}^{T}(\omega,\mathbf{x}_{\perp})}{\omega_{1}^{2}-\left(\omega_{-}+i\eta\right)^{2}} \tag{III.79}$$

$$\chi_{\mathsf{EB}}\left(\omega_{-},\mathbf{x}_{\perp}\right)=\int_{0}^{\infty}d\omega_{1}\frac{\boldsymbol{\alpha}_{\mathsf{EE}}(\omega,\mathbf{x}_{\perp})\cdot\boldsymbol{\alpha}_{\mathsf{BE}}^{T}(\omega,\mathbf{x}_{\perp})+\boldsymbol{\alpha}_{\mathsf{EB}}(\omega,\mathbf{x}_{\perp})\cdot\boldsymbol{\alpha}_{\mathsf{BB}}^{T}(\omega,\mathbf{x}_{\perp})}{\omega_{1}^{2}-\left(\omega_{-}+i\eta\right)^{2}}. \tag{III.80}$$

Due to the symmetries of the system, all the fields can be expanded in a similar manner

$$\mathbf{O}(\mathbf{x},t)=\int\frac{dk_{||}}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathbf{O}(k_{||},\mathbf{x}_{\perp},\omega)e^{i(k_{||}x_{||}-\omega t)}. \tag{III.81}$$

The expansion of the electric field (and analogously the magnetic field) can be expressed similarly to (III.29) via

$$\mathbf{E}(\mathbf{x},t)=\int d^{2}\mathbf{x}'\int\frac{dk_{||}}{2\pi}\int_{0}^{\infty}\frac{d\omega}{2\pi}\Big[i\omega\mu_{0}\mathbf{G}(k_{||},\mathbf{x}_{\perp},\mathbf{x}'_{\perp},\omega)\cdot\mathbf{j}(k_{||},\mathbf{x}_{\perp},\omega)\,e^{i\left[k_{||}x_{||}-(\omega+vk_{||})t\right]} + \text{c.c}\Big]$$

but because of the mixing of the polarisation and the magnetisation due to the Lorentz boost of the fields, the Green's function now satisfies

$$\begin{aligned}\nabla\times\left[\mu^{-1}(\mathbf{x}_{\perp},\omega_{-})\cdot\nabla\times\mathbf{G}(k_{||},\mathbf{x}_{\perp},\mathbf{x}'_{\perp},\omega)\right]+\frac{\omega^{2}}{c^{2}}\epsilon(\mathbf{x}_{\perp},\omega_{-})\cdot\mathbf{G}(k_{||},\mathbf{x}_{\perp},\mathbf{x}'_{\perp},\omega)\\ +i\omega\big\{\boldsymbol{\chi}_{\mathsf{EB}}(\mathbf{x}_{\perp},\omega)\cdot\nabla\times\mathbf{G}(k_{||},\mathbf{x}_{\perp},\mathbf{x}'_{\perp},\omega)\\ -\nabla\times\left[\boldsymbol{\chi}_{\mathsf{BE}}(\mathbf{x}_{\perp},\omega)\cdot\mathbf{G}(k_{||},\mathbf{x}_{\perp},\mathbf{x}'_{\perp},\omega)\right]\big\}=\mathbb{1}_{3}\delta(\mathbf{x}_{\perp}-\mathbf{x}'_{\perp})\end{aligned} \tag{III.82}$$

where here, $\nabla\equiv ik_{||}\hat{\mathbf{k}_{||}}+\nabla_{\perp}$ and the current is expressed as

$$\mathbf{j}(k_{||},\mathbf{x}_{\perp},\omega)=-i\omega\mathbf{P}_{0}(k_{||},\mathbf{x}_{\perp},\omega)+\nabla\times\mathbf{M}_{0}(k_{||},\mathbf{x}_{\perp},\omega) \tag{III.83}$$

where the undriven polarisation $\mathbf{P}_0$ and magnetisation $\mathbf{M}_0$ are the generalised versions (III.22). The quantisation is performed in a similar fashion to the stationary case in terms a new set of polaritons that satisfy

$$\left[\hat{\mathbf{C}}_\lambda\left(k_{||}, \mathbf{x}_\perp, \gamma\omega\right), \hat{\mathbf{C}}^\dagger_{\lambda'}\left(k'_{||}, \mathbf{x}'_\perp, \gamma\omega'\right)\right] = 2\pi\mathbb{1}_3\delta_{\lambda\lambda'}\delta(k_{||}-k'_{||})\delta(\mathbf{x}_\perp - \mathbf{x}'_\perp)\delta(\omega-\omega'). \tag{III.84}$$

The diagonalised Hamiltonian is expressed in terms of these polaritons as

$$\hat{H} = \sum_{\lambda=\text{e,m}} \int d^2\mathbf{x}_\perp \int_{-\infty}^{\infty} \frac{dk_{||}}{2\pi} \int_0^\infty d\omega\ \hbar\omega_+\ \hat{\mathbf{C}}^\dagger_\lambda\left(k_{||}, \mathbf{x}_\perp, \gamma\omega\right) \cdot \hat{\mathbf{C}}_\lambda\left(k_{||}, \mathbf{x}\perp, \gamma\omega\right). \tag{III.85}$$

The field operators are now expanded in the form

$$\begin{aligned}\hat{\mathbf{O}}(\mathbf{x},t) = \sum_{\lambda=\text{e,m}} \int d^2\mathbf{x}'_\perp \int_{-\infty}^{\infty} \frac{dk_{||}}{2\pi} \int_0^\infty d\omega \\ \left[\mathbf{f}^\lambda_O(k_{||}, \mathbf{x}_\perp, \mathbf{x}'_\perp, \omega)\cdot \hat{\mathbf{C}}_\lambda(k_{||}, \mathbf{x}'_\perp, \gamma\omega) e^{i\left[k_{||}x_{||}-(\omega+vk_{||})t\right]} + \text{h.c}\right]\end{aligned} \tag{III.86}$$

where the frequencies are now Doppler shifted. This concludes the brief overview of the quantisation of a moving half space magneto-dielectric when it is homogeneous in the direction of motion, which, crucially is out of plane with the surface of the dielectric. As we will see later on, if the motion has a component normal to the surface, additional effects occur due to the moving boundary.

In order to quantify the excitation of a dipole placed in the vicinity of this moving medium, perturbation theory is used (see section I.5) where the interaction Hamiltonian is given by the coupling between the electromagnetic field and the medium

$$\hat{H}_I = -i\boldsymbol{\kappa}\cdot\hat{\mathbf{E}}(\mathbf{x}_0,t)\left(\hat{b}e^{-i\omega t} - \hat{b}^\dagger e^{i\omega t}\right) \tag{III.87}$$

where $\boldsymbol{\kappa}$ is constant vector indicating the direction of the dipole moment of the two-level system and $\hat{b}$ and $\hat{b}^\dagger$ are the creation and annihilation operators of (I.30). An additional term is added to the Hamiltonian to account for the energy of the two level system given by $\hat{H}_D = \hbar\omega\hat{b}^\dagger\hat{b}$.

The emission rate is then given in [44] in terms of the electromagnetic Green's function

$$\Gamma_{0\to1} = \frac{2\omega^2}{\hbar}\int_{\omega/v}^{\infty}\frac{dk_{||}}{2\pi}\boldsymbol{\kappa}\cdot\mathbf{G}(-k_{||},\mathbf{x}_{\perp0},\mathbf{x}_{\perp0},-\omega)\cdot\boldsymbol{\kappa}. \tag{III.88}$$

The motion is taken to be in the x-direction, with the z-direction taken to be normal to the surface of the dielectric. After substituting in the expression for the Green's function for the half space magneto-dielectric in terms of the Fresnel coefficients, the full emission rate is eventually given, in the low velocity regime, as

$$\begin{aligned}\Gamma_{0\to1} = \frac{\mu_0\omega^2}{\hbar}\sum_\lambda\int_{\omega/v}^{\infty}\frac{dk_x}{2\pi}\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}\frac{e^{-2\xi z_0}}{\xi} \\ \times\left|\boldsymbol{\kappa}\cdot\hat{\mathbf{e}}_\lambda(-k_x,k_y,i\xi)\right|^2\ \mathrm{Im}[\mathrm{r}_{\lambda\lambda}(-k_x,k_y,-i\xi,-\omega)]\end{aligned} \tag{III.89}$$

where $\mathrm{r}_{\lambda\lambda}$ are the Fresnel reflection coefficients. In order to understand the meaning of this expression, the current author applied this result to a specific material using the definition of the Fresnel coefficients

$$\begin{aligned}\mathrm{r}_{11}(-k_x,k_y,-i\xi,-\omega) &= \frac{\sqrt{\omega^2/c^2\epsilon(\omega)-k_x^2-k_y^2}-i\xi}{\sqrt{\omega^2/c^2\epsilon(\omega)-k_x^2-k_y^2}+i\xi} \\ \mathrm{r}_{22}(-k_x,k_y,-i\xi,-\omega) &= \frac{\sqrt{\omega^2/c^2\epsilon(\omega)-k_x^2-k_y^2}-i\xi\epsilon(\omega)}{\sqrt{\omega^2/c^2\epsilon(\omega)-k_x^2-k_y^2}+i\xi\epsilon(\omega)}.\end{aligned} \tag{III.90}$$

where $\xi = \sqrt{k_x^2+k_y^2-\omega^2/c^2}$ and again $\epsilon(\omega)$ is the relative permittivity. We will also make use of the unit vectors for the s-polarisation

$$\hat{\mathbf{e}}_1(-k_x,k_y,i\xi) = \frac{1}{\sqrt{k_y^2-\xi^2}}\left(k_y\hat{\mathbf{z}}-i\xi\hat{\mathbf{y}}\right) \tag{III.91}$$

and the p-polarisation

$$\hat{\mathbf{e}}_2^{(\pm)}(-k_x,k_y,i\xi) = \frac{c}{\omega}\left[\sqrt{k_y^2-\xi^2}\,\hat{\mathbf{x}}\mp\frac{|k_x|}{\sqrt{k_y^2-\xi^2}}\left(k_y\hat{\mathbf{y}}+i\xi\hat{\mathbf{z}}\right)\right]. \tag{III.92}$$

During the course of this chapter, we shall frequently adopt the simplest case of a lossy dispersive medium, where the dielectric function ϵ is Lorentzian, with a resonant frequency ω_0

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{III.93}$$

and where we assume the arbitrary values $\omega_p = 0.5\ \omega_0$, and damping constant $\gamma = 0.1\ \omega_0$. Figure III.3 shows how the excitation rate of the atom varies with the imaginary part of the permittivity (i.e how much the medium absorbs electromagnetic energy). One may notice that in the absence of absorption of the EM energy within the medium, there is no energy exchange between the moving dielectric and the atom. In this sense, dissipation of electromagnetic energy is a crucial factor for the transfer of energy between two *non-overlapping* bodies (for an atom travelling through a medium, dissipation was not a necessary component). Thus it seems that the dissipation seems to facilitate a "leaking" of the field into the vacuum outside the medium. The excitation rate is peaked for a finite value and gradually decreases as the absorption becomes larger. Furthermore, as can be seen in Figures III.4,III.5 and III.6, the excitation rates for the two level system vary greatly with the frequency of the oscillator, the distance from the dielectric and the velocity of the medium. In general, the excitation will be greatest when the frequency of the oscillator is close to matching the resonances in the dielectric as one might expect, see Figure III.4. The rate of absorption of the oscillator disappears exponentially as it is moved away from the medium, which ties in with the previous discussions of the motion leading to evanescent waves leaking out into the vacuum due to the motion of the dielectric. Conversely, the excitation rate diverges at the boundary, however the model breaks down at this point since one would have to include spatial dispersion that would most likely act to inhibit this excitation rate at short distances. Finally, as one might expect, the rate of excitation is zero when the motion is turned off and increases more or less exponentially as the velocity of the medium increases towards that of the speed or light. Note that unlike the results of section II.3, the presence of the two-level system here, breaks the symmetry. It would be interesting to calculate the Poynting vector in this case and see if it is indeed non-zero.

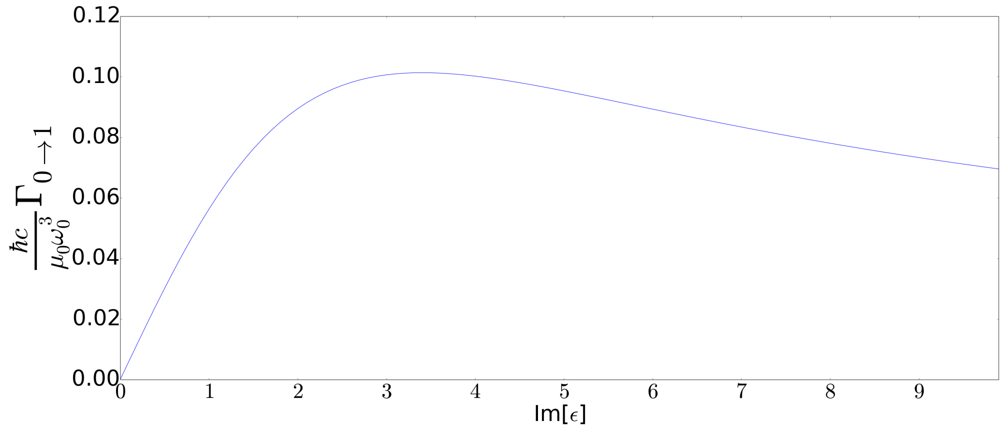

Figure III.3: Excitation rate (III.89) of a two level system moving at a *constant* velocity v relative to half space dielectric as a function of the imaginary part of its permittivity Im$\left[\epsilon(\omega)\right]$ where the Lorentzian permitivity $\epsilon(\omega)$ is given by (III.93) and ω_0 is the resonant frequency of the dielectric. In the absence of any dissipation of electromagnetic energy by the medium (i.e. when Im$[\epsilon(\omega)] = 0$), the excitation rate of the atom is zero meaning that there is no energy exchange between the moving dielectric and the quantum probe. The excitation rate is peaked around Im$[\epsilon(\omega)] \approx 3 - 4$ in this case and dies off gradually as the absorption increases.

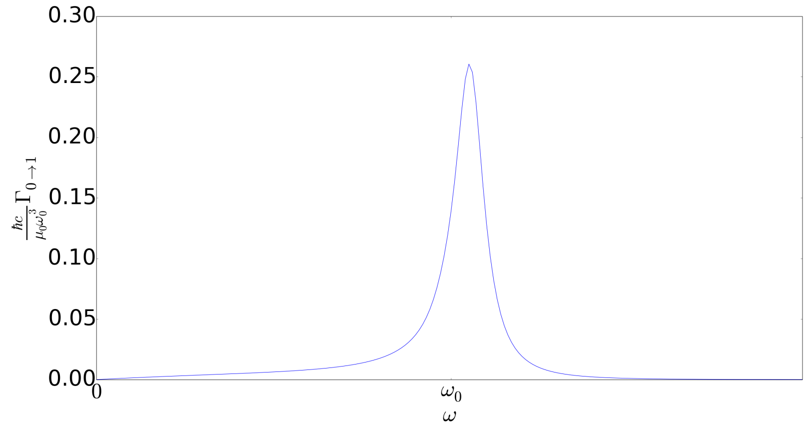

Figure III.4: Variation with respect to frequency ω of absorption rate (III.89) of two level system moving at constant velocity v relative to half space dielectric with the Lorentzian dielectric contant $\epsilon(\omega)$ of (III.93) where ω_0 is the resonant frequency of the dielectric. Unlike the absorptionless case, the resonances in the material play an important role, as can be seen in the increases excitation rate at these resonant frequencies.

III.3.3 Macroscopic QED for non-uniformly moving media

We have considered how to adapt macroscopic QED when the medium moves with a constant velocity but we are primarily interested in the case when the motion is no longer uniform. How does one treat the macroscopic QED model in this case? During the rest of this chapter, the work of the present author will provide

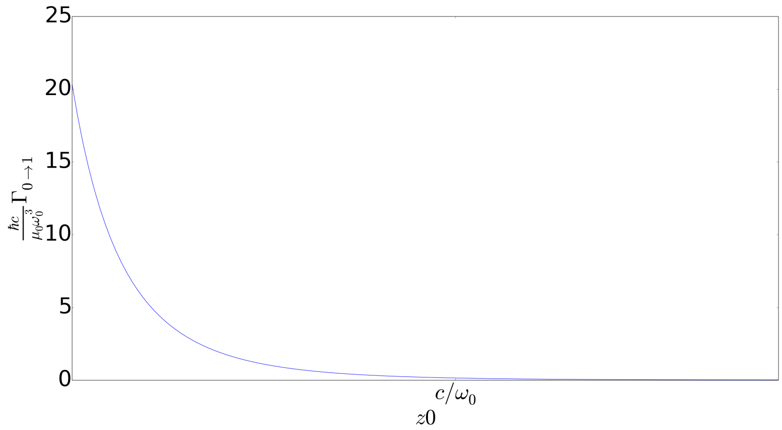

Figure III.5: Variation of the absorption rate (III.89) of two level system moving at constant velocity v relative to half space dielectric with respect to the distance from the dielectric z_0, where the Lorentzian dielectric contant $\epsilon(\omega)$ of (III.93) where ω_0 is the resonant frequency of the dielectric. The excitation rate diverges as the two-level system approaches the dielectric. However, here the model breaks down and in reality spatial dispersion must be taken into account at such distances which should conspire to remove this divergence.

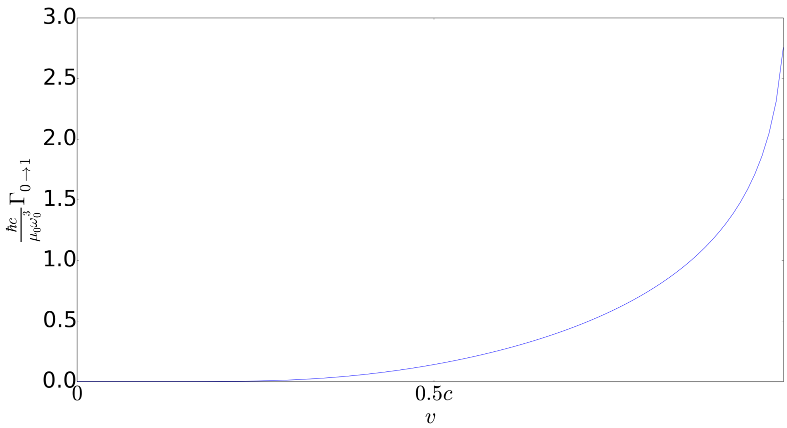

Figure III.6: Variation with respect to the velocity v of absorption rate (III.89) of two level system moving at the *constant* velocity v relative to half space dielectric with the Lorentzian dielectric constant $\epsilon(\omega)$ of (III.93) where ω_0 is the resonant frequency of the dielectric. The excitation rate dies off gradually as the velocity approaches zero until the absorptionless case. It increases more or less exponentially as the velocity of the mediium tends to that of the speed of light, however at those velocities this model would also be invalid as the medium would most likely no longer be modeled by this simple permittivity.

some answers to this question.

The ability to derive exact solutions to the equations of motion in the case of constant motion relies on the fact that the velocity does not depend on time. For example, for a fixed velocity, expression (III.73) indicates a Doppler shift in the frequency where $\omega \to \gamma(\omega - \mathbf{v}\cdot\mathbf{k})$. But imagine that every time interval δt, this velocity changes, so then will the Doppler shift, and thus is seems that field will have a time dependent frequency which in itself indicate difficulties further down the line. For a more rigorous assessment of these difficulties, consider the low velocity approximation to the constant velocity Lagrangian of section III.3.2 (ignoring relativistic terms such as the Lorentz factor) where we now set $\mathbf{v} \to \mathbf{v}(t)$. Through the use of the Euler-Lagrange equations (see e.g. [49]), we obtain the equations of motion for the oscillator bath

$$\left[\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)^2 + \omega^2\right]\mathbf{X}_\omega(\mathbf{x},t) = \alpha_b(\omega)\,\mathbf{E}(\mathbf{x},t) \tag{III.94}$$

and for the field

$$\nabla\times\nabla\times\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}. \tag{III.95}$$

For simplicity we have ignored the polarisation mixing term $\mathbf{B}\cdot\int_0^\infty d\omega\,\boldsymbol{\alpha}_{\boldsymbol{B}}(\omega,\mathbf{x},t)\cdot\mathbf{X}_\omega$, concentrating on only the electric response of the material. In order to see how the time dependence of the medium affects the material parameters, we solve equations (III.94) and (III.95). The solutions to (III.94) are given by

$$\mathbf{X}_\omega(\mathbf{x},t) = \alpha_b(\omega)\int d^3x'\int\limits_{-\infty}^{\infty} dt'\;G_{X_\omega}(\mathbf{x},\mathbf{x}',t,t')\,\mathbf{E}(\mathbf{x},t) + \mathbf{X}_\omega^H(\mathbf{x},t) \tag{III.96}$$

where the $\mathbf{X}_\omega^H(\mathbf{x},t)$ are the homogeneous solutions to (III.94) and are of the general form

$$\mathbf{X}_\omega^H(\mathbf{x},t) = \mathbf{h}_\omega\left(\mathbf{x} - \int_{t_0}^{t}\mathbf{v}(t')\,dt'\right)e^{-i\omega t} + c.c \tag{III.97}$$

where $\mathbf{h}_\omega$ are arbitrary functions of position and the lower limit in the integral, t_0 is the initial time of the evolution of the system where the amplitudes are simply $\mathbf{h}_\omega(\mathbf{x})$. In the quantum theory, these amplitudes will become the creation and annihilation operators of the quantum fields in the same way as the theory developed in [84]. The oscillator Green's function $G_{X_\omega}(\mathbf{x},\mathbf{x}',t,t')$ in (III.96) satisfies

$$\left[\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)^2 + \omega^2\right]G_{X_\omega}(\mathbf{x},\mathbf{x}',t,t') = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta(t-t') \tag{III.98}$$

the retarded solution (zero for $t < t'$) to which is given by

$$G_{X_\omega}(\mathbf{x},\mathbf{x}',t,t') = \frac{1}{\omega}\theta(t-t')\sin[\omega(t-t')]\,\delta^{(3)}\left(\mathbf{x} - \int_{t'}^{t}\mathbf{v}(t_1)\,dt_1\right). \tag{III.99}$$

Substituting (III.99) into (III.96) gives an expression for the amplitudes of the oscillator field in terms of the field $\mathbf{E}$ and the arbitrary amplitudes $\mathbf{h}_\omega$,

$$\mathbf{X}_\omega(\mathbf{x},t) = \frac{\alpha_b(\omega)}{\omega}\int_{-\infty}^{t} dt' \sin[\omega(t-t')]\,\mathbf{E}\left(\mathbf{x} - \int_{t'}^{t}\mathbf{v}(t_1)\,dt_1, t'\right) + \mathbf{X}_\omega^H(\mathbf{x},t). \tag{III.100}$$

The meaning of this integral expression is simply that the oscillator amplitudes at $\mathbf{x}$ no longer depend solely on the strength of the field at $\mathbf{x}$ (as it does in [84]) but also on the previous positions of the moving oscillator for times t' *before* t. In the stationary, or constant velocity case, although the oscillator amplitudes do depend on the field amplitudes at past times, they are confined to one point in space. The motion of the material thus leads to a non–local response, due to the fact that energy dissipated from the electromagnetic field is carried away from the point where it was absorbed. The wave equation for the electric field is found through substituting expression (III.100) into (III.13), and combining the resultant expression with (III.95)

$$\nabla\times\nabla\times\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int d^3x' \int_{-\infty}^{\infty} dt'\,\chi(\mathbf{x},\mathbf{x}',t,t')\,\mathbf{E}(\mathbf{x}',t') = -\mu_0\mathbf{j}(\mathbf{x},t) \tag{III.101}$$

where the non-local effective susceptibility of the system χ is given by

$$\chi(\mathbf{x},\mathbf{x}',t,t') = \frac{1}{\epsilon_0}\int_0^\infty d\omega\frac{\alpha_b^2(\omega)}{\omega}\,\delta^{(3)}\left(\mathbf{x} - \int_{t'}^{t}\mathbf{v}(t_1)\,dt_1 - \mathbf{x}'\right)\theta(t-t')\sin[\omega(t-t')]. \tag{III.102}$$

The theta function in (III.102) ensures that the effective susceptibility is non-zero only for times past, enforcing causality and consequently also the Kramers–Kronig relations. In the limit $v \to 0$, the wave equation (III.101) reduces to the usual wave equation in stationary dispersive media (see e.g. [48]), and (III.102) reduces to the local expression $\chi(\mathbf{x},\mathbf{x}',t,t') = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\chi(t-t')$, where $\chi(t-t')$ is the Fourier transform of $\epsilon_b(\omega)-1$. The source of the electromagnetic field $\mathbf{j}(\mathbf{x},t)$ that appears in the wave equation (III.95) depends on the arbitrary functions $\mathbf{h}_\omega$ and is given by the non-local expression [1]

$$\mathbf{j}(\mathbf{x},t) = \int_0^\infty d\omega\, \alpha_b(\omega)\, \frac{\partial^2}{\partial t^2}\left[\mathbf{h}_\omega\left(\mathbf{x} - \int_{t_0}^{t} \mathbf{v}(t')\, dt'\right) e^{-i\omega t} + c.c\right]. \tag{III.103}$$

This current has the same meaning as in [84]: the un-driven part of the motion of the bath of oscillators is the source of the electromagnetic field in an absorbing medium. In [84], the $\mathbf{h}_\omega$ are related to the amplitude of the current $\mathbf{j}$ in a local manner. Here, the relative motion of the medium $\mathbf{v}$ means that the current at $\mathbf{x}$ now depends on the value of $\mathbf{h}_\omega$ at the point $\mathbf{x} - \int_{t_0}^{t}\mathbf{v}(t')\,dt'$. In analogy to [84], the solution to (III.101) written in terms of the electromagnetic Green's function is

$$\mathbf{E}(\mathbf{x_1},t_1) = -\mu_0 \int d^3\mathbf{x}_2 \int_{-\infty}^{\infty} dt_2\, \mathbf{G}(\mathbf{x_1},\mathbf{x_2},t_1,t_2)\cdot \mathbf{j}(\mathbf{x_2},t_2) \tag{III.104}$$

where $\mathbf{G}$ is a bi-tensor[2] satisfying

$$\begin{aligned}&\nabla\times\nabla\times\mathbf{G}(\mathbf{x_1},\mathbf{x_2},t_1,t_2) + \frac{1}{c^2}\frac{\partial^2}{\partial t_1^2}\mathbf{G}(\mathbf{x_1},\mathbf{x_2},t_1,t_2)\\ &+\frac{1}{c^2}\frac{\partial^2}{\partial t_1^2}\int_{-\infty}^{t_1} dt_3\,\chi(t_1,t_3)\,\mathbf{G}\left(\mathbf{x_1} - \int_{t_3}^{t_1}\mathbf{v}(t')\,dt', \mathbf{x_2}, t_1, t_2\right) = \mathbb{1}_3\delta^{(3)}(\mathbf{x_1}-\mathbf{x_2})\,\delta(t_1-t_2).\end{aligned} \tag{III.105}$$

and $\chi(t,t')$ is equal to (III.102) without the spatial delta function. In the cases treated in [84], the difficulty of finding analytic expressions for the Green function resides in the complexity of the geometry. In this case though, even in homogeneous media, the integro-differential equation above presents great difficulties due to its non-locality. Yet in a few simple cases, some progress can be made. Given the spatial homogeneity of the medium in this case, (III.105) can by Fourier

[1] Note that here the source term $\mathbf{j}$ does not have units of electric current.

[2] A tensorial function of two positions

transformed in space using the expansion of the Green's function

$$\mathbf{G}\left(\mathbf{x}_1-\mathbf{x}_2,t_1,t_2\right)=\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\int\limits_{-\infty}^{\infty}d\omega_1\int\limits_{-\infty}^{\infty}d\omega_2\,\mathbf{G}\left(\mathbf{k},\omega_1,\omega_2\right)e^{i[\mathbf{k}.(\mathbf{x}_1-\mathbf{x}_2)-\omega_1t_1-\omega_2t_2]} \tag{III.106}$$

where the Fourier components satisfy the integral equation

$$\mathbf{k}\times\mathbf{k}\times\mathbf{G}\left(\mathbf{k},\Omega,\Omega_2\right)+\frac{\Omega^2}{c^2}\mathbf{G}\left(\mathbf{k},\Omega,\Omega_2\right)+\frac{\Omega^2}{c^2}\int\limits_{-\infty}^{\infty}\frac{d\Omega_1}{2\pi}\chi\left(\mathbf{k},\Omega,\Omega_1\right)\mathbf{G}\left(\mathbf{k},\Omega_1,\Omega_2\right)$$
$$=-2\pi\mathbb{1}_3\delta(\Omega-\Omega_2) \tag{III.107}$$

and where the kernel of the integral represents the Fourier transformed susceptibility of the system and is given by

$$\chi\left(\mathbf{k},\Omega,\Omega_1\right)=\int\limits_{-\infty}^{\infty}dt\int\limits_{-\infty}^{t}dt_1\,\chi\left(t,t_1\right)e^{-i\mathbf{k}.\int_{t_1}^{t}\mathbf{v}(t')dt'}e^{i\Omega t}e^{-i\Omega_1t_1}. \tag{III.108}$$

For the case of constant velocity $\mathbf{v}(t)=\mathbf{v}$, the wave equation (III.105) simplifies to that given in [44] (again, ignoring polarisation mixing) where the Doppler shifted frequency $\Omega-\mathbf{v}\cdot\mathbf{k}$ appears in the argument of the susceptibility,

$$\left\{\mathbf{k}\otimes\mathbf{k}-\left[k^2-\frac{\Omega^2}{c^2}\left(1+\chi\left(\Omega-\mathbf{k}\cdot\mathbf{v}\right)\right)\right]\mathbb{1}_3\right\}\cdot\mathbf{G}\left(\mathbf{k},\Omega,\Omega_2\right)=-2\pi\delta(\Omega-\Omega_2)\mathbb{1}_3. \tag{III.109}$$

The Green function is then equal to the inverse of the square bracketed matrix on the left times $2\pi\mu_0\delta(\Omega-\Omega_2)$. In other words, the constant motion of the dielectric, gives rise to a new effective permittivity in which the frequency response is shifted by $-\mathbf{k}\cdot\mathbf{v}$.

In the slightly more complicated case of time dependent oscillatory motion in the z-direction, with frequency ν, ($\mathbf{v}(t)=z_0\nu\sin(\nu t)\hat{\mathbf{z}}$, where z_0 is the maximum displacement), the integral over the velocity in the susceptibility (III.108) becomes

$$\mathbf{k}\cdot\int_{t_1}^{t}\mathbf{v}\left(t'\right)dt'=k_z z_0[\cos(\nu t_1)-\cos(\nu t)]. \tag{III.110}$$

The susceptibility (III.108) can then be written as

$$\chi\left(\mathbf{k},\Omega,\Omega_1\right)=2\pi\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}i^{n-m}J_n(k_z z_0)J_m(k_z z_0)\delta(\Omega-\Omega_1+(m+n)\nu)\chi(\Omega_1-n\nu). \tag{III.111}$$

where we applied the generating function for Bessel functions [92]

$$e^{iz\cos\theta}=\sum_{n=-\infty}^{\infty}i^n J_n\left(z\right)e^{in\theta} \tag{III.112}$$

with J_n as the Bessel functions of the first kind. Expression (III.111) demonstrates a coupling between the frequencies of the field arising from the oscillatory motion, occurring in discrete multiples of the oscillation frequency ν. It is interesting to note that the summand vanishes when $k_z z_0$ equals a zero of a Bessel function, which for a fixed $\mathbf{k}$ and Ω excludes the coupling between those frequencies $\Omega_1-(n+m)\nu$ where $J_{n,m}(k_z z_0)=0$. In the limiting case when the argument of the Bessel functions, $k_z z_0$, becomes very large, the material behaves like the vacuum since the Bessel functions all tend to zero. For very small $k_x z_0$, only the $m=n=0$ survives and the material behaves as if it were stationary. This occurs for very large wavelengths or in the limit of a vanishing displacement amplitude.

The above brief examination demonstrates that for most cases of interest it is difficult to find exact solutions to the equations of motion. This is of course also true for any other time dependencies of the system (for example time dependent permittivities.) In order to establish quantitative results in general, proceeding via perturbation theory is a viable option and one we will explore in the next section.

III.4 Time-dependent Dielectrics

The purpose if this section is to consider the effects general time dependencies of media have on the quantum vacuum. This time dependence could either be due to motion (as briefly explored in the previous section), or the fact that the dielectric function is explicitly time dependent (perhaps a non–linear response to an applied field). This section is heavily based on a publication written by the author [2] as a part of this doctoral work. We start by further extending macroscopic QED in order to describe media exhibiting time dependencies (either via their motion or an explicit time-dependence of the permittivity). We then apply time dependent perturbation theory (explained in more detail in section III.4.2) to find the excitation of the medium and field when initially prepared in the ground state. Although such "dynamic Casimir"–type effects have been already well studied [93], the theory typically does not include the effects of dispersion and dissipation, which we include here within the full generality of macroscopic QED. Due to the inclusion of these effects we must emphasise again that these absorbing systems are described in terms of *polaritons* (for a description, see section III.2.4) and therefore that our analysis does not predict photon excitation, but is instead concerned with the excitation rate of these polaritons within the material, which we remind the reader are functions of ω and $\boldsymbol{k}$ since they are combinations of exctiations of the material and of the electromagnetic field as described in Section III.2. The excitations we find do not obey a dispersion relation, but are strongest in well defined regions of ω–$\boldsymbol{k}$ space, and possess a time dependent electric field able to excite a detector embedded within the material. These polaritons are also associated with propagating radiation that can be measured by a photon detector placed outside of the material. To illustrate this connection we give an example of a model detection process in section III.4.5.

First, we will explore the classical Lagrangian and Hamiltonian necessary to describe a dielectric that has time dependent properties, or is in motion. Using time–dependent perturbation theory, we then calculate the excitation rate of pairs of polaritons due to the time dependence of the medium. From the perspective of macroscopic QED time dependent material properties and material motion contribute differently to the Hamiltonian, even though one might expect a moving spatial distribution of permittivity to be indistinguishable from a moving medium. After developing the general theory we treat three illustrative cases (see Figure III.7 for a schematic): (a) an infinite homogeneous medium performing a linear oscillatory motion; (b) an infinite homogeneous medium where the permittivity oscillates in time; and (c) a 'bump' in the permittivity moving uniformly through an otherwise homogeneous dielectric. The final example is inspired by the recent optical experiments investigating laboratory analogues of Hawking radiation [29, 30].

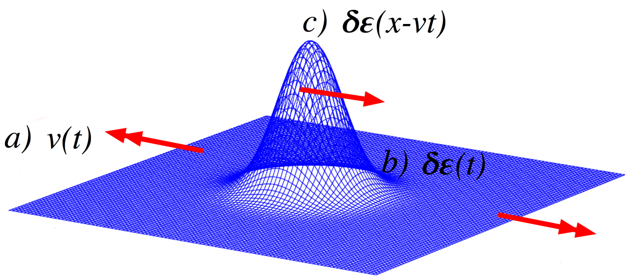

Figure III.7: Three illustrative applications of macroscopic QED to time dependent media: a) an infinite homogeneous medium oscillates at velocity $\mathbf{v}(t) = v(t)\hat{\mathbf{z}}$ around a fixed position; b) the permittivity of a uniform medium oscillates around a fixed value, $\epsilon(\omega, t) = \epsilon_b(\omega) + \delta\epsilon(\omega, t)$; c) a space dependent change to the permittivity $\delta\epsilon(\omega, x, t) = \delta\epsilon(x - vt, \omega)$, moves at a constant velocity $\mathbf{v} = v\hat{\mathbf{z}}$ through an otherwise uniform medium of permittivity $\epsilon_b(\omega)$, emitting radiation as it moves.

We begin by constructing the classical theory of the electromagnetic field interacting with a moving or time dependent material exhibiting dispersion and dissipation. The Lagrangian of [84] is extended in a similar fashion to [44,85], accounting for non-relativistic motion and an arbitrary time dependence of the permittivity profile. The Lagrangian density is written in terms of dynamical variables of the electromagnetic field, the vector potential $\mathbf{A}$ and the scalar potential ϕ, related to the electric and magnetic field strengths by $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ respectively. The dissipation of electromagnetic energy is again provided by the coupling of the electromagnetic field to the field of harmonic oscillators $\mathbf{X}_\omega(\mathbf{x}, t)$ present throughout space, with every possible natural frequency ω.

III.4.1 Lagrangian density

The Lagrangian density for our system is,

$$\mathscr{L} = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 - c^2\mathbf{B}^2\right) + \mathbf{E}\cdot\int\limits_0^\infty d\omega\ \alpha\left(\omega,\mathbf{x},t\right)\mathbf{X}_\omega$$
$$+\mathbf{B}\cdot\int\limits_0^\infty d\omega\ \boldsymbol{\alpha_B}\left(\omega,\mathbf{x},t\right)\mathbf{X}_\omega + \frac{1}{2}\int\limits_0^\infty d\omega\left[\left(\dot{\mathbf{X}}_\omega + \left(\mathbf{v}(\mathbf{x},t)\cdot\nabla\right)\mathbf{X}_\omega\right)^2 - \omega^2\mathbf{X}_\omega^2\right] \tag{III.113}$$

which is valid to first order in the local velocity of the material $\mathbf{v}(\mathbf{x},t)$ and assumes for simplicity that $\mu = 1$. The coupling constant $\boldsymbol{\alpha_B}(\omega,\mathbf{x},t)$ is given in terms of α by $\boldsymbol{\alpha_B}\left(\omega,\mathbf{x},t\right) = -\mathbf{v}\left(\mathbf{x},t\right)\times\alpha\left(\mathbf{x},\omega,t\right)\mathbb{1}_3$. To motivate (III.113) note that for isotropic media the local polarization of the medium is related to the reservoir of harmonic oscillators by the formula [44, 84, 85]

$$\mathbf{P}(\mathbf{x},t) = \int_0^\infty d\omega\,\alpha(\omega,\mathbf{x},t)\mathbf{X}_\omega(\mathbf{x},t) \tag{III.114}$$

with $\alpha(\omega,\mathbf{x},t) = \sqrt{2\omega\epsilon_0\mathrm{Im}[\epsilon(\omega,\mathbf{x},t)]/\pi}$. When the material is in motion with local velocity $\mathbf{v}(\mathbf{x},t)$ this local polarization appears partly as a magnetization (a phenomenon we shall sometimes refer to as polarization mixing), which to first order in velocity equals

$$\mathbf{M}(\mathbf{x},t) = -\mathbf{v}(\mathbf{x},t)\times\mathbf{P} = \int_0^\infty d\omega\,\boldsymbol{\alpha}_B(\omega,\mathbf{x},t)\cdot\mathbf{X}_\omega(\mathbf{x},t). \tag{III.115}$$

It is thus clear that the first three terms of the Lagrangian density (III.113) equal the free space Lagrangian density $\mathscr{L}_\mathrm{F} = (\epsilon_0/2)(\mathbf{E}^2 - c^2\mathbf{B}^2)$, minus the electric dipole interaction energy density $-\mathbf{E}\cdot\mathbf{P}$, and the magnetic dipole interaction energy density $-\mathbf{B}\cdot\mathbf{M}$. The final term is the Lagrangian density for a field of harmonic oscillators $\mathscr{L}_\mathrm{B} = (1/2)\int_0^\infty d\omega[\dot{\mathbf{X}}_\omega^2 - \omega^2\mathbf{X}_\omega^2]$, with the rest frame coordinates $\mathbf{x}', t'$ transformed into the laboratory frame such that $t' = t$ and $\mathbf{x}' = \mathbf{x}'(\mathbf{x},t)$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\boldsymbol{\nabla} \tag{III.116}$$

where $\dot{\mathbf{x}}' = \mathbf{v}(\mathbf{x}, t)$. Although we have set $\mu = 1$, this amounts to ignoring the second oscillator bath that appears in [44, 84, 85], and can be re–introduced with no fundamental modification to our results. The coupling between the field of oscillators and the electromagnetic field is mediated via a term that is proportional to a quantity α which in this case is a scalar quantity that depends on space, time, and frequency. Here we treat the time dependence of the medium as a perturbation to an otherwise isotropic, time independent background: $\alpha\,(\omega, \mathbf{x}, t) = \alpha_b\,(\mathbf{x}, \omega) + \delta\alpha\,(\omega, \mathbf{x}, t)$. The background permittivity of the medium $\epsilon_b(\mathbf{x}, \omega)$ is related to this time independent coupling term by the expression given in [84]: $\alpha_b\,(\mathbf{x}, \omega) = \sqrt{2\omega\epsilon_0 \mathrm{Im}[\epsilon_b\,(\mathbf{x}, \omega)]/\pi}$. Notice that setting $\delta\alpha\,(\omega, \mathbf{x}, t) \to 0$ and $\boldsymbol{\alpha_B}(\omega, \mathbf{x}, t) \to 0$ recovers the situation explored in the previous section starting from the oscillator equations of motion (III.94).

III.4.2 Polariton Excitation Rate of the Quantum Vacuum

The Hamiltonian of the system is defined using (I.37) in terms of the canonical momenta and the Lagrangian where the canonical momentum of the electromagnetic field $\mathbf{\Pi_A}$ and of the oscillator field $\Pi_{\mathbf{X}_\omega}$ are now found to be

$$\begin{aligned}
\Pi_{\mathbf{A}} &= -\epsilon_0 \mathbf{E} - \int_0^\infty d\omega\ \alpha\,(\omega, \mathbf{x}, t)\,\mathbf{X}_\omega \\
\Pi_{\mathbf{X}_\omega} &= \dot{\mathbf{X}}_\omega + (\mathbf{v}\cdot\nabla)\,\mathbf{X}_\omega
\end{aligned} \tag{III.117}$$

where the scalar potential ϕ is again not a dynamical variable.

Because we treat the time dependence of the dielectric as having a small effect on the total field, we divide this Hamiltonian up into two parts as in Section I.5 where the time-independent Hamiltonian H_0 is now the full Hamiltonian of

Section III.2 which we give here in terms of the canonical field amplitudes

$$
\begin{aligned}
H_0 = \frac{1}{2}\int d^3\mathbf{x}\bigg[\frac{1}{\epsilon_0}\bigg(\mathbf{\Pi_A} + \int_0^\infty d\omega\; \alpha_b\left(\mathbf{x},\omega\right)\mathbf{X}_\omega\bigg)^2 \\
+ \frac{1}{\mu_0}\left(\nabla\times\mathbf{A}\right)^2 + \int_0^\infty d\omega\left(\mathbf{\Pi}^2_{\mathbf{X}_\omega} + \omega^2\mathbf{X}^2_\omega\right)\bigg]
\end{aligned}
\tag{III.118}
$$

and the interaction Hamiltonian H_I models the motion and time dependence of the dielectric, and is given by,

$$
\begin{aligned}
H_I = \int d^3\mathbf{x}\int_0^\infty d\omega\bigg[-\Pi_{\mathbf{X}_\omega}\cdot\left(\mathbf{v}\cdot\nabla\right)\mathbf{X}_\omega + \mathbf{B}\cdot\boldsymbol{\alpha_B}\left(\omega,\mathbf{x},t\right)\cdot\mathbf{X}_\omega \\
+ \mathbf{E}\cdot\mathbf{X}_\omega\delta\alpha\left(\omega,\mathbf{x},t\right)\bigg].
\end{aligned}
\tag{III.119}
$$

This interaction Hamiltonian contains three terms: the first accounts for the fact that the bath of oscillators is in motion; the second is due to the mixing of polarizations by the motion; and the third term is due to the time dependence of the permittivity, which could—for example—be due to the motion of the boundaries or simply the time-variation of the permittivity of a stationary object.

As in Section I.5, we work in the interaction picture where the full Hamiltonian is given by $\hat{H} = \hat{H}_0 + \hat{H}_I$. With the time dependence of the operators being generated by $\hat{H}_0$, the quantisation is carried out as in Section III.2, but it is somewhat simpler since we ignore the magnetic oscillator bath $\mathbf{Y}_\omega$ and set $\mu(\mathbf{x},\omega)\to 1$ and the canonical operators are expanded as in (III.55) and (III.56) after dropping the λ subscripts. The stationary Hamiltonian is thus diagonalised in the slightly simpler form

$$
\hat{H}_0 = \int d^3\mathbf{x}\int_0^\infty d\omega\;\hbar\omega\;\hat{\mathbf{C}}\left(\mathbf{x},\omega\right)\cdot\hat{\mathbf{C}}^\dagger\left(\mathbf{x},\omega\right)
\tag{III.120}
$$

where the creation and annihilation operators satisfy the new commutation relations

$$
\begin{aligned}
\left[\hat{\mathbf{C}}\left(\mathbf{x},\omega\right),\hat{\mathbf{C}}^\dagger\left(\mathbf{x}',\omega'\right)\right] &= \mathbb{1}_3\,\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(\omega-\omega'\right) \\
\left[\hat{\mathbf{C}}\left(\mathbf{x},\omega\right),\hat{\mathbf{C}}\left(\mathbf{x}',\omega'\right)\right] &= 0.
\end{aligned}
\tag{III.121}
$$

Note that the only difference to (III.54) is the lack of the subscript λ. We now proceed to apply the methods of perturbation theory used in Section I.5 to our system. The interaction Hamiltonian will contain only terms quadratic in the canonical variables, as such, once expanded in terms of the creation and annihilation operators that diagonalise the non-interacting Hamiltonian, it may be written in terms of time dependant coefficients as

$$\hat{H}_I(t) = \int d^3\mathbf{x} \int d^3\mathbf{x}' \int_0^\infty d\omega \int_0^\infty d\omega' \times$$
$$\vartheta_{11}^{mn}\left(\mathbf{x},\mathbf{x}',\omega,\omega',t\right)\hat{C}_m\left(\mathbf{x},\omega\right)\hat{C}_n\left(\mathbf{x}',\omega'\right) + \vartheta_{12}^{mn}\left(\mathbf{x},\mathbf{x}',\omega,\omega',t\right)\hat{C}_m\left(\mathbf{x},\omega\right)\hat{C}_n^\dagger\left(\mathbf{x}',\omega'\right)$$
$$+ \vartheta_{21}^{mn}\left(\mathbf{x},\mathbf{x}',\omega,\omega',t\right)\hat{C}_m^\dagger\left(\mathbf{x},\omega\right)\hat{C}_n\left(\mathbf{x}',\omega'\right) + \vartheta_{22}^{mn}\left(\mathbf{x},\mathbf{x}',\omega,\omega',t\right)\hat{C}_m^\dagger\left(\mathbf{x},\omega\right)\hat{C}_n^\dagger\left(\mathbf{x}',\omega'\right) \tag{III.122}$$

where the coefficients ϑ_{ij} can be found by substituting the expansion of the fields (III.55) into the expression for the interaction Hamiltonian (III.120). Making use of the expansion (III.120), the Baker-Hausdorff identity

$$e^{i\hat{H}_0 t_1/\hbar}\hat{H}_I\left(t_1\right)e^{-i\hat{H}_0 t_1/\hbar} = \hat{H}_I(t_1) + \frac{it_1}{\hbar}\left[\hat{H}_0, \hat{H}_I(t_1)\right] + \frac{1}{2!}\left(\frac{it_1}{\hbar}\right)^2\left[\hat{H}_0,\left[\hat{H}_0, \hat{H}_I(t_1)\right]\right] + ... \tag{III.123}$$

and the commutation relations (III.121), one finds

$$e^{i\hat{H}_0 t_1/\hbar}\hat{H}_I\left(t_1\right)e^{-i\hat{H}_0 t_1/\hbar} = \int d^3\mathbf{x} \int d^3\mathbf{x}' \int_0^\infty d\omega \int_0^\infty d\omega' \times \vartheta_{22}^{mn}\left(\mathbf{x},\mathbf{x}',\omega,\omega',t_1\right)e^{i(\omega+\omega')t_1}\hat{C}_m^\dagger\left(\mathbf{x},\omega\right)\hat{C}_n^\dagger\left(\mathbf{x}',\omega'\right). \tag{III.124}$$

Thus, the full time-dependent wave function is approximated by

$$\left|\psi\left(t\right)\right\rangle \approx e^{-i\hat{H}_0 t/\hbar}\left|\psi_I\left(0\right)\right\rangle - \frac{i}{\hbar}e^{-i\hat{H}_0 t/\hbar}\int d^3\mathbf{x} \int d^3\mathbf{x}' \int_0^\infty d\omega \int_0^\infty d\omega' \int_{t_0}^t dt_1 \times$$
$$\vartheta_{22}^{mn}\left(\mathbf{x},\mathbf{x}',\omega,\omega',t_1\right)e^{i(\omega+\omega')t_1}\hat{C}_m^\dagger\left(\mathbf{x},\omega\right)\hat{C}_n^\dagger\left(\mathbf{x}',\omega'\right)\left|\psi_I\left(0\right)\right\rangle. \tag{III.125}$$

In the absence of any time dependence, we take the system to be prepared in its ground state $|0\rangle$ (defined as the state where $\hat{\boldsymbol{C}}_{\omega}|0\rangle = 0$). The interaction Hamiltonian $\hat{H}_I$ is a combination of squares of field operators, which will lead to the creation of pairs of polaritons. We thus represent the wave function of the system as

$$\begin{gathered}|\psi(t)\rangle = |0\rangle + \sum_{m,n} \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2 \int_0^{\infty} d\omega_1 \int_0^{\infty} d\omega_2 \times \\ \xi_{mn}\left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t\right) \hat{C}_m^{\dagger}\left(\mathbf{x}_1, \omega_1\right) \hat{C}_n^{\dagger}\left(\mathbf{x}_2, \omega_2\right) |0\rangle\end{gathered} \tag{III.126}$$

where the expansion coefficient obeys $\xi_{mn}\left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t\right) = \xi_{nm}\left(\mathbf{x_2}, \mathbf{x_1}, \omega_2, \omega_1, t\right)$, in accordance with bosonic exchange symmetry. The physical meaning of ξ_{mn} is as the probability amplitude for the creation of a pair of current excitations in the material, located at positions $\mathbf{x}_1$ and $\mathbf{x}_2$, oscillating at frequencies ω_1 and ω_2, and pointing in the m and n directions. Inserting (III.126) into the Schrodinger equation $\partial|\psi\rangle/\partial t = -(i/\hbar)\hat{H}_I|\psi\rangle$, we find the rate of change of the expansion coefficient is given by

$$\dot{\xi}_{mn}\left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t\right) = -\frac{i}{2\hbar}\langle 0|\hat{C}_m(\mathbf{x}_1, \omega_1)\hat{C}_n(\mathbf{x}_2, \omega_2)\hat{H}_I|0\rangle \tag{III.127}$$

Due to the form of the interaction Hamiltonian (III.119), the rate of change of the rank-2 tensor ξ_{mn} separates into the sum of two parts,

$$\dot{\xi}_{mn}\left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t\right) = \dot{\xi}_{mn}^{(a)}\left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t\right) + \dot{\xi}_{mn}^{(b)}\left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t\right). \tag{III.128}$$

The first term $\xi_{mn}^{(a)}$ arises from the first two terms in the interaction Hamiltonian (III.119) and represents the effect of the motion of the medium, both through the movement of energy within the bath of oscillators, and the mixing of the electromagnetic field polarization between reference frames. Note that although in general the coupling term $\boldsymbol{\alpha}_B\left(\omega, \mathbf{x}, t\right)$ depends upon the time dependence of the material $\alpha\left(\omega, \mathbf{x}, t\right)$ as well as the motion, in what follows we shall only treat particular cases where $\boldsymbol{\alpha_B}\left(\omega, \mathbf{x}, t\right) = -\mathbf{v}\left(\mathbf{x}, t\right) \times \alpha_b\left(\mathbf{x}, \omega\right) \mathbb{1}_3$ (i.e. the motion of a homogeneous medium). The second term, $\xi_{mn}^{(b)}$ comes from the final term in (III.119), and represents the effect of any time dependence in the permittivity. Evaluating (III.127) by substituting the expansion of the operators (III.55) and (III.56) into the interaction Hamiltonian and using the commutation relations (III.121), the dyadic form the rate of change of the two parts to the expansion coefficient (III.128) is found to be

$$\dot{\boldsymbol{\xi}}^{(a)}\left(\mathbf{x_1},\mathbf{x_2},\omega_1,\omega_2,t\right)=-\frac{i}{2\hbar}e^{i(\omega_1+\omega_2)t}\int d^3\mathbf{x}\int_0^\infty d\omega$$
$$\times\left[\mathbf{f}^{\dagger}_{\mathbf{\Pi X}}\left(\mathbf{x},\mathbf{x_1},\omega,\omega_1\right)\cdot\left(\mathbf{v}\cdot\nabla\right)\mathbf{f}^{*}_{\mathbf{X}}\left(\mathbf{x},\mathbf{x_2},\omega,\omega_2\right)\right.$$
$$\left.+\,\mathbf{f}^{\dagger}_{\mathbf{B}}\left(\mathbf{x},\mathbf{x_1},\omega_1\right)\cdot\boldsymbol{\alpha}_B\left(\omega,\mathbf{x},t\right)\cdot\mathbf{f}^{*}_{\mathbf{X}}\left(\mathbf{x},\mathbf{x_2},\omega,\omega_2\right)\right]+1\leftrightarrow2 \tag{III.129}$$

and

$$\dot{\boldsymbol{\xi}}^{(b)}\left(\mathbf{x_1},\mathbf{x_2},\omega_1,\omega_2,t\right)=-\frac{i}{2\hbar}e^{i(\omega_1+\omega_2)t}\int d^3\mathbf{x}\int_0^\infty d\omega$$
$$\times\left[\delta\alpha\left(\omega,\mathbf{x},t\right)\mathbf{f}^{\dagger}_{\mathbf{E}}\left(\mathbf{x},\mathbf{x_1},\omega_1\right)\cdot\mathbf{f}^{*}_{\mathbf{X}}\left(\mathbf{x},\mathbf{x_2},\omega,\omega_2\right)+1\leftrightarrow2\right] \tag{III.130}$$

where the notation '$1\leftrightarrow2$' indicates the repetition of the preceding expression but with the two particles interchanged (which in the above expressions involves both taking the transpose and swapping subscripts 1 and 2). It is evident from expressions (III.129) and (III.130) that the rate of polariton excitation is in general quite different for time dependent and moving media. For instance, a time dependent permittivity profile constructed to appear as a moving material would have $\boldsymbol{\xi}^{(a)}=0$, whereas true motion has in general both non–zero $\boldsymbol{\xi}^{(a)}$ and $\boldsymbol{\xi}^{(b)}$. We shall now evaluate the polariton excitation rates (III.129) and (III.130) for the three cases shown in Figure III.7.

III.4.3 Motion of a Homogeneous Dielectric

The first example we investigate is the polariton emission rate within an infinite homogeneous medium performing an oscillatory motion. Due to the lack of boundaries and the translational invariance (and thus a time-independent permittivity), the second term in (III.128), $\dot{\boldsymbol{\xi}}^{(b)}$ is equal to zero, and the probability amplitude for emitting a pair of polaritons is then equal to the integral of $\dot{\boldsymbol{\xi}}^{(a)}$ over the time interval $t \in [-T/2, T/2]$ (it is assumed that the interaction Hamiltonian is 'turned on' at the initial time $-T/2$). The absolute value squared of the result divided by T gives us the average net excitation rate of polariton pairs over the time interval T. Over a very long time interval $T \to \infty$ this is given by

$$\Gamma = \int d^3\mathbf{x_1} \int d^3\mathbf{x_2} \int\limits_0^\infty d\omega_1 \int\limits_0^\infty d\omega_2 \sum_{m,n} \lim_{T\to\infty} \frac{1}{T} \left| \int\limits_{-T/2}^{T/2} dt\ \dot{\xi}^{(a)}_{mn} \left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t\right) \right|^2 . \tag{III.131}$$

The emission rate given by (III.131) is valid even for spatially dependant velocities $\mathbf{v}$, as occurs e.g. for rotating bodies. However here we make the simplification that the velocity does not depend on position, in which case—as is evident from (III.129)—the time dependence of $\dot{\xi}^{(a)}$ can be factored out from the spatial dependence. For example, in the case of constant velocity the time dependence of (III.129) is given by the factor $\exp(i(\omega_1 + \omega_2)t)$ so that

$$\lim_{T\to\infty} \frac{1}{T} \left| \int\limits_{-T/2}^{T/2} dt\ e^{i(\omega_1+\omega_2)t} \right|^2 = 2\pi\delta\left(\omega_1 + \omega_2\right). \tag{III.132}$$

The emission rate (III.131) is then reduced to

$$\Gamma = 2\pi \int d^3\mathbf{x_1} \int d^3\mathbf{x_2} \int\limits_0^\infty d\omega_1 \int\limits_0^\infty d\omega_2 \sum_{m,n} \left| \mathcal{A}^{(a)}_{mn} \left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2\right) \right|^2 \delta\left(\omega_1 + \omega_2\right) \tag{III.133}$$

where, from expression (III.129), we have factored out the time-dependence via the substitution $\dot{\xi}^{(a)}_{mn} \left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t\right) = \mathcal{A}^{(a)}_{mn} \left(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2\right) \exp\left(i\left(\omega_1 + \omega_2\right)t\right)$. The argument of the delta function is only zero at the one point $\omega_1 = \omega_2 = 0$, where the integrand is zero. This leaves,

$$\Gamma = 0 \tag{III.134}$$

as expected.

When the velocity is time dependent then the emission rate Γ will be non–zero. Consider some time dependent velocity $\mathbf{v}(t) = v(t)\hat{\mathbf{z}}$. In order to analyse the emission rate as a function of frequency and wave–vector, we work in terms of the Fourier transform of $\dot{\boldsymbol{\xi}}^{(a)}$,

$$\begin{aligned}\dot{\boldsymbol{\xi}}^{(a)}(\mathbf{x_1}, \mathbf{x_2}, \omega_1, \omega_2, t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \dot{\boldsymbol{\xi}}^{(a)}(\mathbf{k}, \omega_1, \omega_2, t)\, e^{i\mathbf{k}.(\mathbf{x_1}-\mathbf{x_2})} \\ &= v(t)\, e^{i(\omega_1+\omega_2)t} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \boldsymbol{\mathcal{A}}(\mathbf{k}, \omega_1, \omega_2)\, e^{i\mathbf{k}.(\mathbf{x_1}-\mathbf{x_2})}\end{aligned} \tag{III.135}$$

where the quantity $\boldsymbol{\mathcal{A}}(\mathbf{k}, \omega_1, \omega_2)$ is given by

$$\boldsymbol{\mathcal{A}}(\mathbf{k}, \omega_1, \omega_2) = \frac{1}{2}\left[\boldsymbol{\mathcal{B}}(\mathbf{k}, \omega_1, \omega_2) + \boldsymbol{\mathcal{B}}(-\mathbf{k}, \omega_2, \omega_1)\right] \tag{III.136}$$

and $\boldsymbol{\mathcal{B}}(\mathbf{k}, \omega_1, \omega_2)$ is given by

$$\begin{aligned}\boldsymbol{\mathcal{B}}(\mathbf{k}, \omega_1, \omega_2) = k_z \sqrt{\epsilon_I(\omega_1)\,\epsilon_I(\omega_2)}\, a(\omega_1, \omega_2)\, \mathbf{G}^{\dagger}(\mathbf{k}, \omega_1) \cdot \mathbf{G}^{*}(-\mathbf{k}, \omega_2) \\ + b(\omega_1, \omega_2)\, \mathbf{G}^{*}(-\mathbf{k}, \omega_2) + c(\omega_1, \omega_2)\, \mathbf{G}^{\dagger}(\mathbf{k}, \omega_1) \\ + d(\omega_1, \omega_2) \frac{1}{k_z}\left[\mathbf{G}^{\dagger}(\mathbf{k}, \omega_1) \times \mathbf{k}\right] \cdot (\hat{\mathbf{z}} \times \mathbb{1}_3) \cdot \left[\chi(\omega_2) \frac{\omega_2^2}{c^2} \mathbf{G}^{*}(-\mathbf{k}, \omega_2) + \mathbb{1}_3\right]\end{aligned}$$

with the coefficients within this quantity given as

$$\begin{aligned}a(\omega_1, \omega_2) &= \frac{1}{\pi c^4} \omega_1^3 \omega_2^2 \left[\frac{\epsilon^*(\omega_1)}{\omega_1^2 - (\omega_2 - i\omega_1 0^+)^2} + \frac{\epsilon^*(\omega_2)}{\omega_2^2 - (\omega_1 - i\omega_2 0^+)^2}\right] \\ b(\omega_1, \omega_2) &= \frac{\omega_1}{\pi c^2} \frac{\omega_2^2}{\omega_1^2 - (\omega_2 - i\omega_1 0^+)^2} \\ c(\omega_1, \omega_2) &= \frac{\omega_1}{\pi c^2} \frac{\omega_1^2}{\omega_2^2 - (\omega_1 - i\omega_2 0^+)^2} \\ d(\omega_1, \omega_2) &= \frac{i\omega_1}{\pi c^2}.\end{aligned} \tag{III.137}$$

The Fourier amplitudes of the Green's function $\mathbf{G}(\mathbf{k},\omega)$ are such that $\mathbf{G}\left(\mathbf{x}_1-\mathbf{x}_2,t_1-t_2\right)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\mathbf{G}\left(\mathbf{k},\omega_1\right)e^{i[\mathbf{k}.(\mathbf{x}_1-\mathbf{x}_2)-\omega_1 t_1]}$ and can be found to be

$$\mathbf{G}\left(\mathbf{k},\omega_1\right)=\frac{\mathbf{k}\otimes\mathbf{k}-\epsilon_b\left(\omega_1\right)\left(\omega_1/c\right)^2\mathbb{1}_3}{\epsilon_b(\omega_1)\left(\omega_1/c\right)^2\left[\epsilon_b\left(\omega_1\right)\omega_1^2/c^2-k^2\right]}. \tag{III.138}$$

The vacuum expansion of the Green's function (III.106) is recovered by taken the limit $\epsilon_b(\omega)\rightarrow 1$ in expression (III.138) at which a delta function appears that sets the dispersion relation between the wave-vector and the frequency that is *absent* in this absorbing medium case. It is the denominator in (III.138) that is responsible for the appearance of this delta function in this limit and it is plotted for a Lorentzian dielectric in Figure III.4.3. This two dimensional dispersion relation is the origin of the wave-vector *and* frequency dependence of the polariton creation and annihilation operators discussed previously.

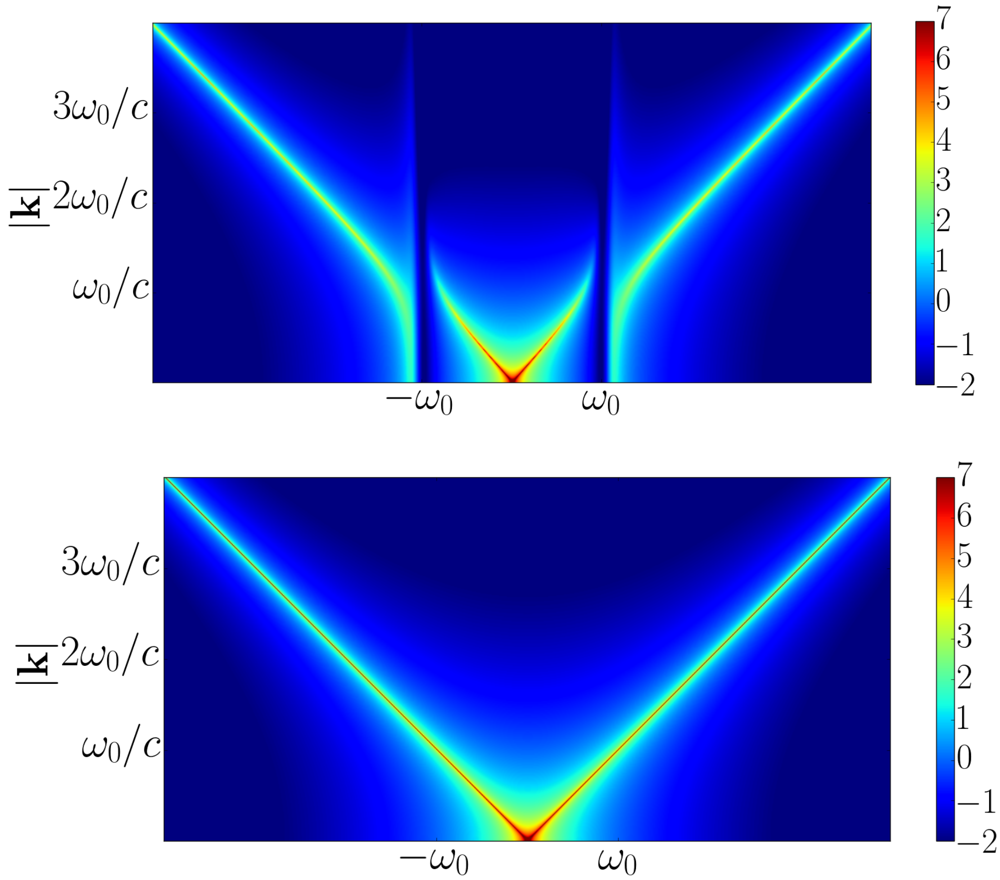

Figure III.8: The Green's tensor (III.138) appears in the various expressions for the excitation rates. Many of the features of these rates have their origin in the features of this term. Plot a) represents the value of the fraction $1/\epsilon(\omega)[c^2k^2 - \omega^2\epsilon(\omega)]$ that appears in the expression for the Green's tensor (III.138). The dielectric function of the material ϵ_b is given by the Lorentzian response (III.93) with a resonance at ω_0. The denominator of (III.138) is largest around frequencies where the two curves satisfying $c^2k^2 - \epsilon^\star(\omega)\omega^2 = 0$. The two vertical lines at $\omega = \pm\omega_0$ represent the resonances of the material at which point absorption within the medium is highest and propagation of light is maximally reduced. In the vacuum limit $\epsilon_b \to 1$ plotted in b), the two vertical lines at $\omega = \pm\omega_0$ disappear and the remaining dispersion curves become straight lines representing the usual vacuum dispersion relation $|\mathbf{k}| = \pm\omega$. These combination of these features encode the notion that polaritons represent a mixture of electromagnetic excitations and material excitations.

The amplitude $\mathcal{A}(\mathbf{k}, \omega_1, \omega_2)$ is proportional to the probability amplitude for exciting a pair of polaritons with wave-vectors $\mathbf{k}$ and $-\mathbf{k}$ and frequencies ω_1 and ω_2 from the ground state, due to the motion of the medium.

We take the particular case: $v(t) = z_0\nu\cos(\nu t)$, where z_0 is the maximum displacement from the mean position. Taking the absolute value squared of (III.135) and integrating with respect to time, the equivalent of (III.132) now equals

$$\lim_{T\to\infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} dt\; v\,(t)\, e^{i(\omega_1+\omega_2)t} \right|^2 = \frac{\pi z_0^2 \nu^2}{2} \left[\delta\left(\omega_1+\omega_2-\nu\right)+\delta\left(\omega_1+\omega_2+\nu\right)\right] \tag{III.139}$$

the proof of which is given in Appendix B. While the second term on the right of (III.139) fails to contribute to the emission rate (owing to ω_1, ω_2 and ν all being positive), the first term does. Combining (III.135) and (III.139) with the expression for the net rate of excitation (III.133) gives the emission rate per unit volume V

$$\frac{\Gamma}{V} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_0^\infty d\omega\; \rho(\mathbf{k},\omega) \tag{III.140}$$

where the 'spectral density' for the emission of polariton pairs ρ is equal to

$$\rho(\mathbf{k},\omega) = \frac{\pi z_0^2 \nu^2}{2} \theta(\nu-\omega) \sum_{m,n} \left|\mathcal{A}_{mn}(\mathbf{k},\omega,\nu-\omega)\right|^2 \tag{III.141}$$

with θ the Heaviside step function. This dimensionless 'spectral density' depends only on the wave vector and frequency since the translational symmetry of this system guarantees momentum conservation for the two polaritons, $\mathbf{k}_1+\mathbf{k}_2=0$, and energy conservation implies that the total energy of the pair of particles must equal that due to the motion: $\hbar\nu=\hbar(\omega_1+\omega_2)$. Note that the time dependence of the velocity sets this relationship between ω_1 and ω_2, and that for a general motion this may be more complicated.

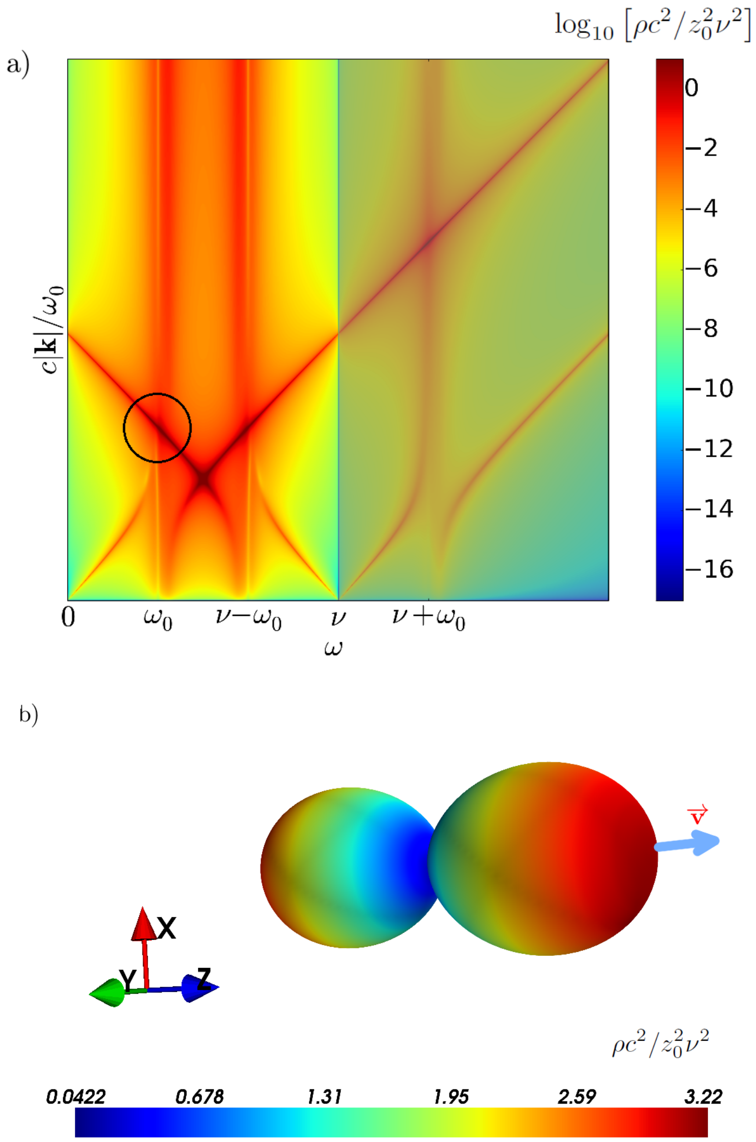

Figure III.9: a): Logarithm of the spectral density of polariton emission $\log_{10}[\rho c^2/z_0^2\nu^2]$, due to the oscillatory motion of a homogeneous dielectric. The dielectric function of the material ϵ_b is given by the Lorentzian response (III.93) with a resonance at ω_0, while the oscillation frequency of the motion is arbitrarily chosen as $\nu = 3\omega_0$ to clearly separate the curves. The quantity ρ is mostly concentrated around frequencies in the vicinity of the resonant frequency of the material ω_0, the shifted resonant frequency $\nu - \omega_0$ as well as along the two dispersion curves satisfying $c^2k^2 - \epsilon^*(\omega)\omega^2 = 0$, and the shifted curve $c^2k^2 - \epsilon^*(\nu - \omega)(\nu - \omega)^2 = 0$. The vertical line at $\omega = \nu$ represents the upper boundary of the area contributing to the emission rate (III.140), due to the Θ function in (III.141); the shaded area to right of this does not contribute to the total emission rate. The black circle points to an example of a region where the shifted dispersion curve intersects with the resonance at ω_0; the emission rate density is much more intense and somewhat spread out around these areas. b): Angular dependence of spectral density $\rho(\mathbf{k})c^2/z_0^2\nu^2$ where the frequency and wave vector magnitude are arbitrarily chosen to be $\omega = \omega_0$ and $|\mathbf{k}| = 2\omega_0$ respectively corresponding to the centre of the black circle in (a). The motion of the dielectric is in the z-direction as indicated by the pale blue arrow. The distance of the surface from the centre of the graph shows the amplitude of the emission rate for varying directions of $\mathbf{k}$. The strongest excitation of polaritons occurs in the direction of motion.

The excitation rate (III.140) evidently scales quadratically with the maximum displacement z_0. However, the dependence on the oscillation frequency ν is somewhat more intricate. As can be seen from the expression for $\boldsymbol{\mathcal{A}}$ (expression (III.136)), as well as Figure III.9a, the largest contribution to the spectral density ρ comes from regions where the frequency and wave vector are close to satisfying the dispersion relations for electromagnetic waves $\omega = c|\mathbf{k}|/\epsilon^{\star}(\omega)$ and $\nu-\omega = \pm c|\mathbf{k}|/\epsilon^{\star}(\nu-\omega)$ and where the frequency matches the resonant frequency of the dielectric ω_0, and the shifted resonance at $\nu - \omega_0$. The largest emission of polaritons occurs in the region close to where the dispersion curves for radiation intersect with the material resonances. These excitations can be measured through detecting the electric field at a point inside or outside the medium. In section III.4.5 we show how the excitation rate of such a detector is related to the quantity ξ_{nm} that appears within ρ.

III.4.4 Time Dependent Permittivities

The second contribution to the emission rate (III.130) $\dot{\boldsymbol{\xi}}^{(b)}$ comes from the time dependence of the material response $\delta\alpha\left(\mathbf{x}, t, \omega\right)$. For moving media, this term can be non–zero either due to moving inhomogeneities or changing boundaries. This term can also be used to model changes in the permittivity due to external forces, such as those in dynamical Casimir experiments [41, 42] or optical analogues of Hawking radiation [29].

Inserting the expansion coefficients listed in equations (III.57) into (III.130), we find the rate of change of the probability amplitude for exciting a pair of polaritons is given by

$$\dot{\boldsymbol{\xi}}^{(b)}\left(\mathbf{x_1},\mathbf{x_2},\omega_1,\omega_2,t\right) = -\frac{i\mu_0\omega_1^2\alpha_b\left(\omega_1\right)}{4\sqrt{\omega_1\omega_2}}e^{i(\omega_1+\omega_2)t}\times$$

$$\left[\mu_0\omega_2^2\alpha_b\left(\omega_2\right)\int_0^\infty d\omega\frac{\alpha_b\left(\omega\right)}{\omega^2-\left(\omega_2-i\omega 0^+\right)^2}\int d^3\mathbf{x}\;\delta\alpha\left(\omega,\mathbf{x},t\right)\;\mathbf{G}^\dagger\left(\mathbf{x},\mathbf{x_1},\omega_1\right)\cdot\mathbf{G}^*\left(\mathbf{x},\mathbf{x_2},\omega_2\right)\right.$$
$$\left.+\,\delta\alpha\left(\omega_2,\mathbf{x_2},t\right)\mathbf{G}^\dagger\left(\mathbf{x_2},\mathbf{x_1},\omega_1\right)\right]+1\leftrightarrow 2 \tag{III.142}$$

We now consider two particular applications of (III.142), firstly where the permittivity of the medium uniformly oscillates as a function of time, and secondly where a travelling pulse-like perturbation $\delta\alpha\left(\mathbf{x}-vt,\omega\right)$ to the permittivity moves through the medium at a uniform velocity.

Emission from a Time Dependent Permittivity

For a time dependent change to the permittivity that takes the form $\delta\alpha\left(\omega,t\right) = \alpha_0\beta\left(\omega\right)\cos\left(\nu t\right)$, the time dependence can be factored out as in the previous section, and the identity (III.139) can be applied. The dimensionless 'spectral density' appearing in the polariton emission rate per unit volume (the analogue of (III.140)) is then found to be

$$\begin{aligned}\rho\left(\mathbf{k},\omega\right) &= \frac{\pi}{2}\int_0^\infty d\omega_2\;\delta\left(\omega+\omega_2-\nu\right)\sum_{m,n}\left|\mathcal{A}_{mn}\left(\mathbf{k},\omega,\omega_2\right)\right|^2 \\ &= \frac{\pi}{2}\sum_{m,n}\left|\mathcal{A}_{mn}\left(\mathbf{k},\omega,\nu-\omega\right)\right|^2\theta(\nu-\omega)\end{aligned} \tag{III.143}$$

where

$$\boldsymbol{\mathcal{A}}\left(\mathbf{k},\omega_1,\omega_2\right) = -\frac{i\mu_0\,\alpha_0\,\omega_1^2\,\alpha_b(\omega_1)}{4\sqrt{\omega_1\omega_2}}\left[\mu_0\,\alpha_b\left(\omega_2\right)\omega_2^2\int_0^\infty d\omega\frac{\alpha_b\left(\omega\right)\beta\left(\omega\right)}{\omega^2-\left(\omega_2-i\omega 0^+\right)^2}\right.$$
$$\left.\times\,\mathbf{G}^\dagger\left(\mathbf{k},\omega_1\right)\cdot\mathbf{G}^*\left(-\mathbf{k},\omega_2\right)+\beta\left(\omega_2\right)\mathbf{G}^\dagger\left(\mathbf{k},\omega_1\right)\right]+1\leftrightarrow 2. \tag{III.144}$$

with '$1 \leftrightarrow 2$' again indicating an interchange of the two particles (which now involves swapping the subscripts 1 and 2, interchanging $\mathbf{k}$ for $-\mathbf{k}$ and taking the transpose). As in the case of the moving dielectric that we just discussed, the emission conserves momentum $\mathbf{k}_1 + \mathbf{k}_2 = 0$ and the energy of the pair of polaritons is taken from the oscillation: $\hbar\nu = \hbar(\omega_1 + \omega_2)$. Equation (III.144) can be further simplified in the particular case where $\beta(\omega) = \alpha_b(\omega)$ (i.e. the change in the permittivity has the same frequency dependent response as the background),

$$\begin{aligned}\boldsymbol{\mathcal{A}}(\mathbf{k}, \omega_1, \omega_2) = &-\frac{i\alpha_0}{2\pi}\sqrt{\mathrm{Im}[\chi_b(\omega_1)]\mathrm{Im}[\chi_b(\omega_2)]} \\ &\frac{\omega_1^2}{c^2}\mathbf{G}^{\dagger}(\mathbf{k}, \omega_1) \cdot \left[\chi_b^{\star}(\omega_2)\frac{\omega_2^2}{c^2}\mathbf{G}^{*}(-\mathbf{k}, \omega_2) + \mathbb{1}_3\right] + 1 \leftrightarrow 2.\end{aligned} \tag{III.145}$$

As is evident from (III.143)—and in similarity to the time dependent velocity investigated in the previous section—the time dependence of $\delta\alpha$ gives rise to emission only for frequencies in the range $\omega_1 > \nu$. The emission rate per unit volume is then given by integrating (III.143) over wave vector and frequency. The frequency and wave vector dependence of the spectral density of polariton emission are plotted in Figure III.10. Due to the lack of any preferred direction in the changing permittivity $\delta\alpha$ the emission is isotropic, but otherwise it is in many respects similar to that of a dielectric in oscillatory motion. However, note that the emission rate is not proportional ν^2 in this case. The case explored here is usually referred to as dynamic-Casimir type effect [34–36, 94] as discussed in the introduction. However, it should be emphasised again that, similarly to the problem discussed in Section III.3.2, the equivalence between moving modies and time dependent dielectrics is a subtle one; for dynamic-Casimir type effects, the results are not the same (see for example [34]). This might be due to there being effects from the first term (III.129) which are usually not taken into account [34–37, 39, 40].

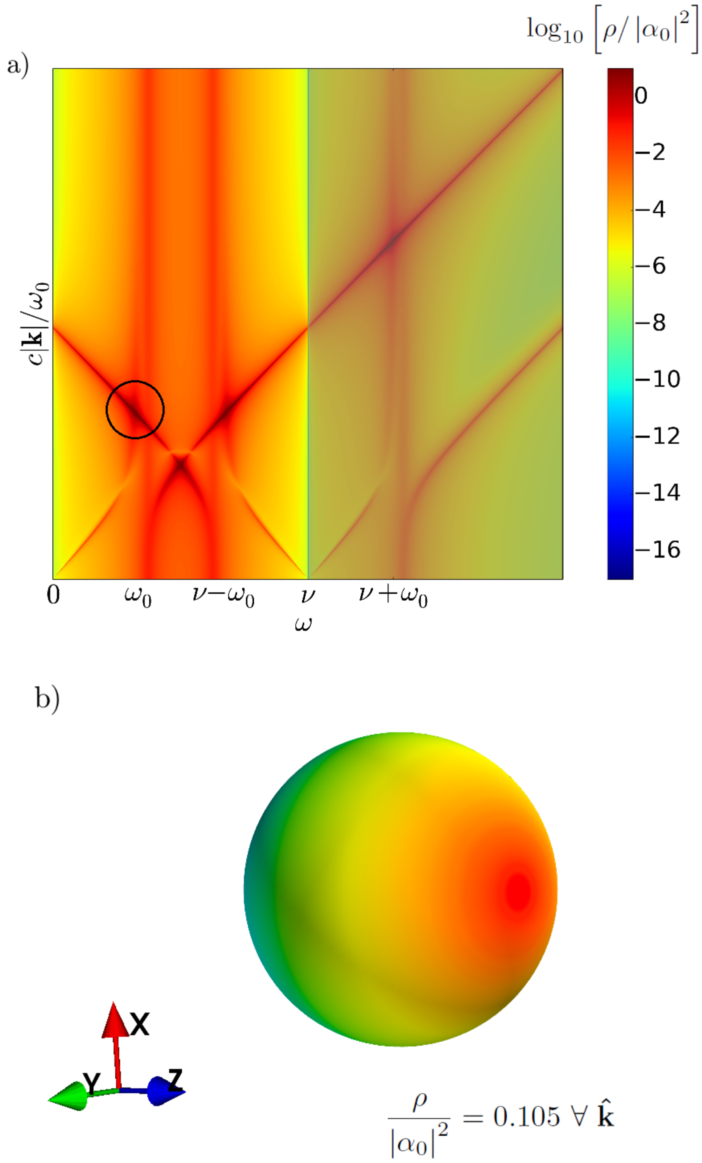

Figure III.10: a): Logarithm of the spectral density of polariton emission $\log_{10}[\rho/|\alpha_0|^2]$ as a function of ω and $c|\mathbf{k}|$, for a dielectric function oscillating as a function of time with frequency ν. The background dielectric function of the material ϵ_b is given by the Lorentzian response (III.93) with a resonance at ω_0, while the oscillation frequency of the material properties is chosen as $\nu = 3\omega_0$ to clearly separate the curves. The regions of high emission occur at the same points as shown for the moving dielectric in figure III.9. b): Angular dependence of spectral density $\rho(\mathbf{k})/|\alpha_0|^2$ where the frequency and wave vector magnitude are arbitrarily chosen to be $\omega = \omega_0$ and $|\mathbf{k}| = 2\omega_0$ respectively corresponding to the centre of the black circle in (a). The colour in this sub-figure only serves to illustrate that the shape is a sphere, indicating isotropic emission.

As a more involved example of macroscopic QED applied to time dependent media, we address the case of a medium through which a perturbation of the refractive index travels at a constant velocity $\mathbf{v} = v\hat{\mathbf{z}}$, producing pairs of polaritons. Recent work [29–31] has established a connection between such a process and an analogue of Hawking radiation [23, 29]. Here we do not emphasize the connection to general relativity, but rather look for a description of such an emission process that fully accounts for the effects of dispersion and dissipation. To the authors' knowledge, previous treatments have not fully accounted for such effects. We assume that the travelling perturbation to the permittivity can be represented in terms of a function $\delta\alpha(\omega, \mathbf{x}, t)$ taking the form

$$\delta\alpha\left(\omega,\mathbf{x},t\right)=\beta\left(\omega\right)\int\frac{d^{3}\mathbf{k}}{\left(2\pi\right)^{3}}f\left(\mathbf{k}\right)e^{i\mathbf{k}.\left(\mathbf{x}-v\hat{\mathbf{z}}t\right)} \tag{III.146}$$

where $f(\mathbf{k})$ is some arbitrary spectral function. Note that unlike the case of the truly moving medium considered above, v is not restricted to non-relativistic velocities. In analogy to the calculations given above we find the following expression for the rate of change of the polariton amplitude,

$$\begin{aligned}\dot{\boldsymbol{\xi}}^{(b)}\left(\mathbf{x_1},\mathbf{x_2},\omega_1,\omega_2,t\right)=&\int\frac{d^3\mathbf{k}_1}{\left(2\pi\right)^3}\int\frac{d^3\mathbf{k}_2}{\left(2\pi\right)^3}\boldsymbol{\mathcal{A}}\left(\mathbf{k}_1,\mathbf{k}_2,\omega_1,\omega_2\right)\\&\times f\left(\mathbf{k}_1+\mathbf{k}_2\right)e^{-i(\mathbf{k}_1+\mathbf{k}_2).\mathbf{v}t}e^{i(\omega_1+\omega_2)t}e^{i\mathbf{k}_1.\mathbf{x_1}}e^{i\mathbf{k}_2.\mathbf{x_2}}\end{aligned} \tag{III.147}$$

where the dyadic $\boldsymbol{\mathcal{A}}$ is given by

$$\begin{aligned}\boldsymbol{\mathcal{A}}(\mathbf{k}_1,&\mathbf{k}_2,\omega_1,\omega_2)=-\frac{i\mu_0\alpha_b(\omega_1)\omega_1^2}{4\sqrt{\omega_1\omega_2}}\mathbf{G}^\dagger(\mathbf{k}_1,\omega_1)\\&\cdot\left[\mu_0\omega_2^2\alpha_b(\omega_2)\int\limits_0^\infty d\omega\frac{\beta(\omega)\alpha_b(\omega)}{\omega^2-(\omega_2-i\omega0^+)^2}\mathbf{G}^\star(\mathbf{k}_2,\omega_2)+\beta(\omega_2)\mathbb{1}_3\right]\end{aligned} \tag{III.148}$$

For simplicity, we again choose the change in permittivity to have the same dispersion as the background material $\beta(\omega) = \alpha_b(\omega)$. Equation (III.148) then reduces to

$$\boldsymbol{\mathcal{A}}\left(\mathbf{k}_1,\mathbf{k}_2,\omega_1,\omega_2\right)=-\frac{i}{2\pi}\sqrt{\mathrm{Im}[\chi\left(\omega_1\right)]\mathrm{Im}[\chi\left(\omega_2\right)]}\frac{\omega_1^2}{c^2}\mathbf{G}^{\dagger}\left(\mathbf{k}_1,\omega_1\right) \cdot\left[\chi_b^{\star}\left(\omega_2\right)\frac{\omega_2^2}{c^2}\mathbf{G}^{*}\left(\mathbf{k}_2,\omega_2\right)+\mathbb{1}_3\right]+1\leftrightarrow 2. \tag{III.149}$$

Defining the net emission rate in terms of a spectral density that is now a function of two wave–vectors and frequency

$$\Gamma=\int_0^{\infty}d\omega\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\int\frac{d^3\mathbf{k}_2}{(2\pi)^3}\,\rho(\mathbf{k}_1,\mathbf{k}_2,\omega)$$

we find ρ to be

$$\rho(\mathbf{k}_1,\mathbf{k}_2,\omega)=2\pi\sum_{m,n}\left|\mathcal{A}_{mn}(\mathbf{k}_1,\mathbf{k}_2,\omega,\mathbf{v}\cdot(\mathbf{k}_1+\mathbf{k}_2)-\omega)\right|^2\left|f(\mathbf{k}_1+\mathbf{k}_2)\right|^2 \times\theta\left[\mathbf{v}\cdot(\mathbf{k}_1+\mathbf{k}_2)-\omega\right] \tag{III.150}$$

where we applied

$$\lim_{T\to\infty}\frac{1}{T}\left|\int_{-T/2}^{T/2}dt\;e^{i(\omega_1+\omega_2-\mathbf{v}\cdot(\mathbf{k}_1+\mathbf{k}_2))t}\right|^2=2\pi\delta\left[\omega_1+\omega_2-\mathbf{v}\cdot(\mathbf{k}_1+\mathbf{k}_2)\right]. \tag{III.151}$$

The expression for ρ has a similar form to that given in the previous sections. However, the lack of translational symmetry (or equivalently the momentum exchanged between the polaritons and the moving perturbation) means that this ‘spectral density’ ρ now depends on two wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, and it now has dimensions of volume squared.

Despite the rather dense notation, the form of (III.150) can be motivated from fairly simple physical considerations: the spatial distribution of the moving perturbation to the permittivity $\delta\alpha$ is determined by the function $f(\mathbf{x})$, which has a corresponding Fourier spectrum $f(\mathbf{k})$. The two polaritons can exchange momentum with the moving perturbation so long as the conservation law $\hbar\mathbf{k}=\hbar(\mathbf{k}_1+\mathbf{k}_2)$ is satisfied. Energy conservation must also be obeyed, with the pulse containing frequencies $\mathbf{v} \cdot \mathbf{k}$ so that $\hbar(\omega_1 + \omega_2) = \hbar\mathbf{v} \cdot \mathbf{k}$. Combining these two conservation laws leads to the constraint $\omega_2 = \mathbf{v} \cdot (\mathbf{k}_1 + \mathbf{k}_2) - \omega_1$, with the rate of pair production being proportional to the amplitude of the relevant Fourier amplitude of the perturbation, $|f(\mathbf{k}_1 + \mathbf{k}_2)|^2$. The θ function in (III.150) ensures that the energy of both of the polaritons is positive, while $|\mathcal{A}_{nm}|^2$ scales the emission rate depending on how close the frequencies ω_1 and ω_2 are to the resonances of the material (where α_b is large), and how close the dispersion relations $|\mathbf{k}_{1,2}|^2 = \epsilon^\star(\omega_{1,2})\omega_{1,2}^2/c^2$ are to being fulfilled (where the Green function $\mathbf{G}$ is large). From (III.149) it is once again evident that the peak pair production will occur when the dispersion relation is fulfilled or the frequency of one or both members of the pair are at a resonance of the material. In particular we note that it is possible for the moving refractive index perturbation to generate pairs of excitations where only one of the pair is a propagating electromagnetic mode. This corresponds to frequencies and wave–vectors where both the material resonance and the dispersion relation are close to being fulfilled.

In figure III.11 we plot the dependence of $\rho(\mathbf{k}_1, \mathbf{k}_2, \omega)$ as a function of both the relative angle between $\mathbf{k}_1$ and $\mathbf{k}_2$, and the magnitude of the modulus of both wave–vectors when $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}$ for a fixed angle $\cos(\theta) = k_z/|\mathbf{k}|$. We take the particular case where the moving perturbation to the permittivity takes the form of a Gaussian,

$$f(\mathbf{k}) = f_0 \exp\left(\frac{\sigma^2}{4}\mathbf{k}^2\right)$$

with the area under the function $f(\mathbf{x})$ equal to f_0. The spectral density for the excitation of polariton pairs (III.150) then becomes

$$\begin{aligned}\rho(\mathbf{k}_1, \mathbf{k}_2, \omega) = 2\pi|f_0|^2 \sum_{m,n} &|\mathcal{A}_{mn}(\mathbf{k}_1, \mathbf{k}_2, \omega, \mathbf{v} \cdot (\mathbf{k}_1 + \mathbf{k}_2) - \omega)|^2 \\ &\exp\left(\frac{\sigma^2}{2}[\mathbf{k}_1 + \mathbf{k}_2]^2\right)\theta(\mathbf{v} \cdot (\mathbf{k}_1 + \mathbf{k}_2) - \omega)\end{aligned} \tag{III.152}$$

It is evident from (III.152) that there is a trade off in the spectral density between the requirement of energy conservation, and the shape of the moving perturbation. The maximum in the Fourier spectrum $f(\mathbf{k}_1 + \mathbf{k}_2)$ in this case occurs when

$\mathbf{k}_1 = -\mathbf{k}_2$, which is where the energy of one of the polaritons $\mathbf{v} \cdot (\mathbf{k}_1 + \mathbf{k}_2) - \omega$ is negative, and thus the spectral density is zero. Meanwhile when $\mathbf{k}_1 = \mathbf{k}_2$, both polaritons can have positive energy, but the exponential factor $\exp(\sigma^2[\mathbf{k}_1 + \mathbf{k}_2]^2/2)$ reduces the spectral density exponentially with the magnitude of $\mathbf{k}_{1,2}$. Spatially sharp, rapidly moving perturbations thus have relatively large emission rates for pairs of polaritons, and these pairs tend to be emitted away from the axis of propagation. This is in qualitative agreement with the findings of e.g. [29] but here derived without the analogy to general relativity. Spatially broad perturbations correspond to large values of σ, and thus small values of $\mathbf{k}_{1,2}$. As the velocity of the perturbation drops to 0, the θ function in (III.152) picks out ever larger magnitudes of $\mathbf{k}_{1,2}$, which are ever further from fulfilling the dispersion relation (i.e. where the Green function $\mathbf{G}(\mathbf{k}, \omega)$ in (III.149) becomes ever smaller). In this zero velocity limit the spectral density thus reduces to zero, as expected.

Figure III.11a confirms the generic behaviour observed in the previous two sections: the emission rate spectral density contains terms that contribute significantly for frequencies and wave-vectors where one of the pairs is close to fulfilling the dispersion relation and/or the condition for resonant material response. Figure III.11b shows the angular dependence of the spectral density of the emission, as the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ is varied for a fixed frequency and equal magnitude of the two wave–vectors. The emission is concentrated in several cone-like regions. These cones correspond to either the shifted dispersion curves or the shifted resonances seen as blurred lines in figure III.11a. Increasing the absorption within the dielectric response has the effect of further blurring the lines of III.11a and of thickening the shells of the cone-like shapes of III.11b. This effectively means that it becomes possible to excite polaritons that lie further from the resonance of the material or the dispersion curve for electromagnetic waves. We emphasize that unlike the previous sections, the choices made in figure III.11 do not show the complete radiation pattern from the moving moving perturbation and only serve to show under what conditions the emission is increased, as well as the directional distribution of the emission.

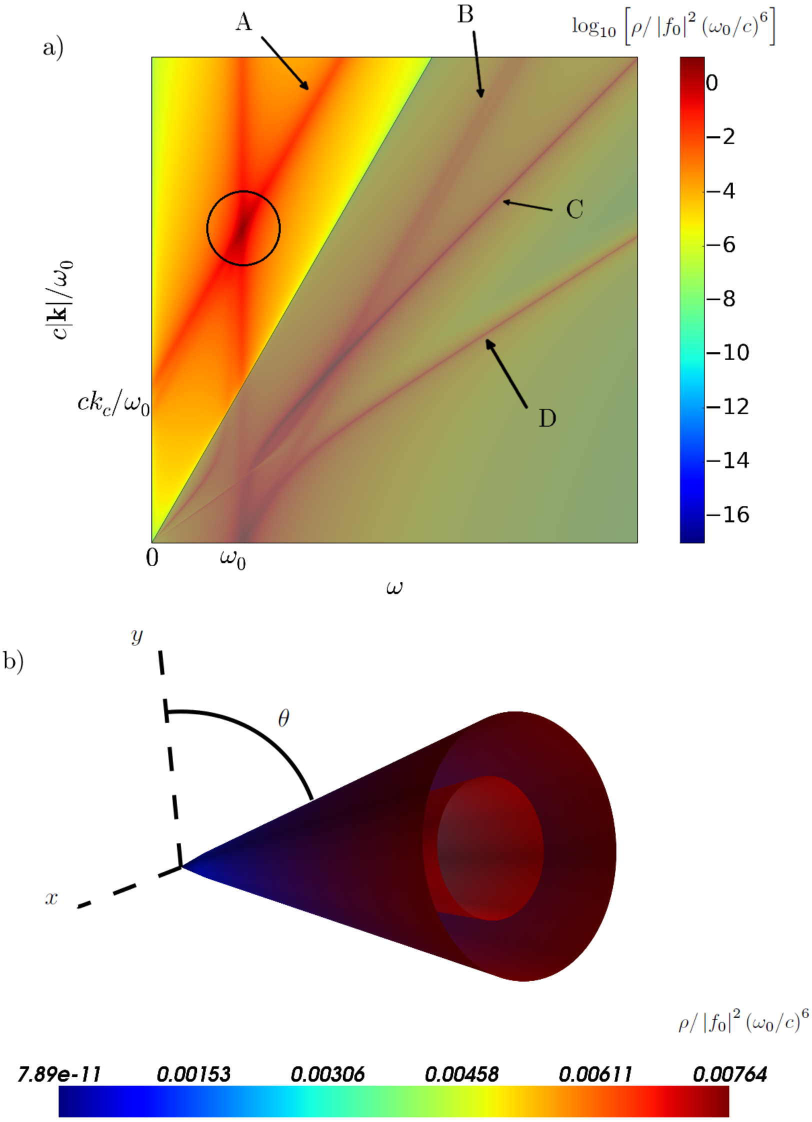

Figure III.11: a): Logarithm of the spectral density for polariton emission $\log_{10}[\rho/|f_0|^2(\omega_0/c)^6]$ for $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}$ at an arbitrary angle given by $\cos(\theta) = k_z/|\mathbf{k}| = 1/\sqrt{3}$. The background dielectric function of the material ϵ_b is given by the Lorentzian response (III.93) with a resonance at ω_0. The velocity is taken to be $v = 0.5c$ and the Gaussian pulse width as $\sigma = 0.1c/\omega_0$. The quantity k_c equals $\sqrt{3}\omega_0/2v$. The spectral density is significant at frequencies in the vicinity of the resonant frequency of the material (vertical line at $\omega = \omega_0$), the shifted resonant frequencies $v(k_z + k_z') \pm \omega_0$ (A and B) and along the dispersion curve satisfying $k^2 - \epsilon^*(\omega_k)\omega_k^2 = 0$ (see for example line D) and the shifted dispersion curve $k^2 - \epsilon^*(\omega_{k,v} - vk_z)(\omega_{k,v} - vk_z)^2 = 0$ (see for example line C). The shaded area to the right of the line $\omega = v(k_z + k_z')$ does not contribute to the total emission rate due to the Θ function in the spectral density (III.152). The spectral density is greatest around intersections between dispersion curves and/or resonances of the material. The black circle points to a particular example of a region where two resonances coincide. b) Angular dependence of the spectral density $\rho/|f_0|^2(\omega_0/c)^6$ for a fixed frequency $\omega_1 = \omega_0$ where the frequency and wave vector magnitude are arbitrarily chosen to be $\omega = \omega_0$ and $|\mathbf{k}| = \omega_0\sqrt{3}$ respectively corresponding to the centre of the black circle in (a), for varying angle between $\mathbf{k}'$ and $\mathbf{k}$. The rate is seen to be negligible for all directions except along the surface of a number of cones traced out along rings of constant inclination angle θ.

It is interesting to compare this effect with those explored in sections II.1 and III.4.4; in particular the time dependent permittivity of section III.4.4 which would typically be called a dynamic–Casimir type effect. In cases of oscillatory motion or material time dependence, pairs of polaritons can be emitted with the total energy of the pair $\hbar\nu$ coming from the frequency of oscillation. The emission from a moving perturbation is of exactly the same form, and from the perspective of the perturbation theory comes from the same 'dynamic Casimir' term (III.130). The only difference is that in this case the energy available for the creation of excitations instead comes from the frequencies $\mathbf{v}\cdot\mathbf{k}$ that are due to the motion of the refractive index perturbation.

III.4.5 Measuring the Excitation Rates

In this section, we consider how the excitation rates are connected to a possible measurement. Consider coupling a small two-level detector of transition frequency Ω to the electromagnetic field, either inside or outside the material. Here, we establish a relationship between the expansion coefficients in (III.128) and the excitation rate of this detector. If the detector is placed at the position $\mathbf{x}_0$ the Hamiltonian gains an extra interaction term (I.56). The electric field is expanded as

$$\hat{\mathbf{E}}\left(\mathbf{x},t\right)=\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\int\limits_0^\infty d\omega\left\{\mathbf{f}_\mathbf{E}\left(\mathbf{k},\omega\right)\cdot\hat{\mathbf{C}}\left(\mathbf{k},\omega\right)e^{i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)}+\text{h.c}\right\}. \tag{III.153}$$

Prior to coupling to the detector we take the initial state of the system to be

$$\left|\psi_0\right\rangle=\left|0\right\rangle_d\otimes$$
$$\left[\left|0\right\rangle+\sum_{m,n}\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\int\limits_0^\infty d\omega_1\int\limits_0^\infty d\omega_2\;\xi_{mn}\left(\mathbf{k},\omega_1,\omega_2,T\right)\hat{C}_m^\dagger\left(\mathbf{k},\omega_1\right)\hat{C}_n^\dagger\left(\mathbf{k},\omega_2\right)\left|0\right\rangle\right] \tag{III.154}$$

where ξ_{mn} is e.g. the expression derived in section II.1, supposing that the time dependence of the medium has been driven for a fixed time T. The coupling of the detector to the electromagnetic field leads to a probability amplitude for the detector to make a transition to an excited state, reducing the number of quanta in the field and medium by one,

$$|\psi(t)\rangle = |\psi_0\rangle + \sum_p \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_0^\infty d\omega\, \xi_p\left(\mathbf{k},\omega,t\right) \hat{C}_p^\dagger\left(\mathbf{k},\omega\right) \hat{a}^\dagger |0\rangle_d \otimes |0\rangle. \tag{III.155}$$

Using the above interaction Hamiltonian to time evolve the system, the rate of change of ξ_p is found to be

$$\frac{\partial \boldsymbol{\xi}(\mathbf{k},\omega,t)}{\partial t} = -\sqrt{\frac{2\hbar}{\Omega}} e^{i\mathbf{k}\cdot\mathbf{x}_0} \int_0^\infty d\omega_1\, \boldsymbol{\kappa}\cdot\mathbf{f_E}\left(\mathbf{k},\omega_1\right)\cdot\boldsymbol{\xi}^{\mathrm{T}}\left(\mathbf{k},\omega_1,\omega,T\right) e^{i(\Omega-\omega_1)t} \tag{III.156}$$

where $\boldsymbol{\xi}^{\mathrm{T}}\left(\mathbf{k},\omega_1,\omega,T\right)$ is defined via expressions (III.129) and (III.130). Integrating over a time $[0,\tau]$ and using the result

$$\begin{aligned}
&\lim_{\tau\to\infty} \frac{e^{i(\omega-\omega')\tau/2}}{\tau} \frac{\sin((\Omega-\omega)\tau/2)}{(\Omega-\omega)/2} \frac{\sin((\Omega-\omega')\tau/2)}{(\Omega-\omega')/2} \\
&= \lim_{\tau\to\infty} \frac{1}{\tau}\left[\pi^2\delta(\omega-\Omega)\delta(\omega'-\Omega) + i\pi \mathrm{P}\left(\frac{\delta(\Omega-\omega)}{\omega'-\Omega} - \frac{\delta(\Omega-\omega')}{\omega-\Omega}\right) + \mathrm{P}\frac{1}{\omega-\Omega}\mathrm{P}\frac{1}{\omega'-\Omega}\right]
\end{aligned} \tag{III.157}$$

(which can be shown using $\lim_{\tau\to\infty}\mathrm{P}[e^{i\tau x}/x] = i\pi\delta(x)$) we obtain the detector's transition rate

$$\begin{aligned}
\Gamma &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_0^\infty d\omega \frac{\left|\boldsymbol{\xi}(\mathbf{k},\omega,\tau)\right|^2}{\tau} \\
&= \frac{2\pi^2\hbar}{\Omega} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_0^\infty d\omega\, \boldsymbol{\kappa}\cdot\mathbf{f_E}(\mathbf{k},\Omega)\cdot \frac{\boldsymbol{\xi}^{\mathrm{T}}(\mathbf{k},\Omega,\omega,T)\cdot\boldsymbol{\xi}^\star(\mathbf{k},\Omega,\omega,T)}{\tau}\cdot\mathbf{f}_{\mathbf{E}}^\dagger(\mathbf{k},\Omega)\cdot\boldsymbol{\kappa}
\end{aligned} \tag{III.158}$$

where we took the limit $\tau\to\infty$, assuming that the ratio T/τ does not diverge or tend to zero. Note that the terms in addition to the product of delta functions in (III.157) do not contribute to (III.158) in the limit $\tau\to\infty$ because they yield terms proportional to something finite times $1/\tau$. It is thus clear that our model detector is excited at a rate proportional to the integral of the square of $\boldsymbol{\xi}$ divided by τ. This square of $\boldsymbol{\xi}$ has an almost identical form to the polariton excitation rate calculated in the previous section, but here it is weighted by the Green's function of the electric field $\mathbf{f}_E$ and the dipole strength $\boldsymbol{\kappa}$. It is worth mentioning that when the detector is embedded in a medium, the excitation rate may diverge due to coupling with the longitudinal part of the electric field. One means to avoid this calculational difficulty is given in [69].

III.4.6 Correlation Functions for Time Dependent dielectrics

The creation of polariton pairs out of the vacuum has the effect of changing the correlations between photons at different points in space. In order to quantify the changes this brings about, in this section we will consider how the electric field at two points in space are correlated for a given time within time-dependant media. Although we are primarily concerned with the vacuum case in this thesis, we will describe the more general thermal case, as it is the zero temperature limit of this that gives us the correlators we seek. The specific time dependence we will consider to illustrate our findings will be that of Section III.4.3, i.e a homogeneous, infinite dielectric oscillating back and forth with a frequency ν. The findings in this section are exposed in a paper written by the author in the process of being published at the time of submission of this work [3].

For time independent magneto-dielectrics for which the model described in III.2 is valid, the Casimir stress for bodies in thermal equilibrium was derived in [95] by computing the thermal correlation functions of the field operator products that appear in the energy-stress tensor. For example, the equal time correlation function $\langle\hat{\mathbf{E}}\otimes\hat{\mathbf{E}}\rangle$ for the electric field at two positions $\mathbf{x}$ and $\mathbf{x}'$ was found to be the usual result that

$$\left\langle\hat{\mathbf{E}}(\mathbf{x},t)\otimes\hat{\mathbf{E}}(\mathbf{x}',t)\right\rangle=\frac{\hbar\mu_0}{\pi}\int\limits_0^\infty d\omega\,\omega^2\coth\left(\frac{\hbar\omega}{2k_BT}\right)\mathrm{Im}\left[\mathbf{G}\left(\mathbf{x},\mathbf{x}',\omega\right)\right] \qquad \text{(III.159)}$$

where T is the temperature of the system. The question of interest here, is how these correlation functions change when the system is subject to some small time dependence, such as as relative motion of dielectrics or a time dependence in the permittivity for example. To first order in perturbation theory, the time dependent two point correlation function of the electric field, or analogously the magnetic field, is written as a sum of the stationary correlation plus a first order correction

$$\left\langle \hat{\mathbf{E}}\left(\mathbf{x},t\right)\otimes\hat{\mathbf{E}}\left(\mathbf{x}',t\right)\right\rangle_{\beta}\approx\left\langle \hat{\mathbf{E}}\left(\mathbf{x},t\right)\otimes\hat{\mathbf{E}}\left(\mathbf{x}',t\right)\right\rangle_{\beta}^{(0)}+\left\langle \hat{\mathbf{E}}\left(\mathbf{x},t\right)\otimes\hat{\mathbf{E}}\left(\mathbf{x}',t\right)\right\rangle_{\beta}^{(1)} \quad \text{(III.160)}$$

where the first order correction is found from the thermal expectation of the commutation of the product of the operators with the Hamiltonian integrated over all past times during which the perturbation was active,

$$\left\langle \hat{\mathbf{E}}\left(\mathbf{x},t\right)\otimes\hat{\mathbf{E}}\left(\mathbf{x}',t\right)\right\rangle_{\beta}^{(1)}=-\frac{i}{\hbar}\int_{-\infty}^{t}dt'\,\left\langle\left[\hat{\mathbf{E}}\left(\mathbf{x},t\right)\otimes\hat{\mathbf{E}}\left(\mathbf{x}',t\right),\hat{H}_{I}\left(t'\right)\right]\right\rangle_{\beta}. \quad \text{(III.161)}$$

The interaction Hamiltonian $\hat{H}_I$ is a combination of squares of field operators so this commutation will lead to thermal averages of combinations of four creation or annihilation operators. As usual, combinations with unequal numbers of creation and annihilation operators will vanish.

Now, in order to understand the form the remaining terms will take, consider as an example, the 1-Dimensional harmonic oscillator of displacement x from its equilibrium position at some finite temperature. The Lagrangian for this system is given by (I.25), and the Hamiltonian by (I.28), and we therefore again use the quantum mechanical operators (I.31) but this time in the Heisenberg picture where the time dependence of the operators is such that

$$\frac{\partial}{\partial t}\hat{x}(t)=\frac{i}{\hbar}\left[\hat{H},\hat{x}(t)\right] \quad \text{(III.162)}$$

which leads to the expression

$$\hat{x}(t)=\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}e^{-i\omega t}+\hat{a}^{\dagger}e^{i\omega t}\right). \quad \text{(III.163)}$$

In order to arrive at (III.159) in [95], it was necessary to modify the vacuum polariton correlators using

$$\begin{aligned}\left\langle \hat{\mathbf{C}}^{\dagger}\left(\mathbf{x},\omega\right)\otimes\hat{\mathbf{C}}\left(\mathbf{x}',\omega'\right)\right\rangle_{\beta} &= N\left(\omega\right)\mathbb{1}_{3}\,\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(\omega-\omega'\right)\\ \left\langle \hat{\mathbf{C}}\left(\mathbf{x},\omega\right)\otimes\hat{\mathbf{C}}^{\dagger}\left(\mathbf{x}',\omega'\right)\right\rangle_{\beta} &= \left[N\left(\omega\right)+1\right]\mathbb{1}_{3}\,\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(\omega-\omega'\right)\end{aligned} \tag{III.164}$$

where $N\left(\omega\right) = \left[e^{\hbar\omega\beta}-1\right]^{-1}$ is the Bose-Einstein distribution, $\beta = 1/k_B T$ is the thermodynamic β and again T is the temperature of the system. To see where this comes from, consider the analogue of (III.164) here, the thermal average

$$\left\langle \hat{a}\hat{a}^{\dagger}\right\rangle_{\beta} = \frac{\text{Tr}\left[e^{-\beta\hat{H}}\hat{a}\hat{a}^{\dagger}\right]}{\text{Tr}\left[e^{-\beta\hat{H}}\right]} \tag{III.165}$$

where the Hamiltonian operator is $\hat{H} = \hbar\omega(\hat{a}^{\dagger}\hat{a}+1/2)$. In this case, it is simple to work out this average,

$$\left\langle \hat{a}\hat{a}^{\dagger}\right\rangle_{\beta} = \left[\sum_{n=0}^{\infty}\left\langle n\right| e^{-\beta\hat{H}}\left|n\right\rangle\right]^{-1}\sum_{n=0}^{\infty}\left\langle n\right| e^{-\beta\hat{H}}\hat{a}\hat{a}^{\dagger}\left|n\right\rangle . \tag{III.166}$$

Using the usual results that $\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$ and $\hat{a}^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$, we can write this as

$$\left\langle \hat{a}\hat{a}^{\dagger}\right\rangle_{\beta} = \left[\sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}\right]^{-1}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}\left(n+1\right). \tag{III.167}$$

By taking the derivative with respect to frequency of the geometric series

$$\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{1}{1-e^{-\beta\hbar\omega}} \tag{III.168}$$

one can write the identity

$$\begin{aligned}\frac{\partial}{\partial\omega}\left(\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}\right) &= \frac{\partial}{\partial\omega}\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)\\ \sum_{n=0}^{\infty} n\, e^{-\beta\hbar\omega n} &= \frac{e^{-\beta\hbar\omega}}{\left(1-e^{-\beta\hbar\omega}\right)^{2}}.\end{aligned} \tag{III.169}$$

From which, we can write

$$\begin{aligned}\left\langle \hat{a}\hat{a}^{\dagger}\right\rangle_{\beta} &= \frac{1-e^{-\beta\hbar\omega}}{e^{-\beta\hbar\omega/2}} \times e^{-\beta\hbar\omega/2}\left[\frac{e^{-\beta\hbar\omega}}{\left(1-e^{-\beta\hbar\omega}\right)^{2}}+\frac{1}{1-e^{-\beta\hbar\omega}}\right] \\ &= N\left(\omega\right)+1.\end{aligned} \tag{III.170}$$

This allows us to suppose that in the case of a continuum of oscillators, when the frequency is the same, the thermal average $\langle \hat{\mathbf{C}}_{\lambda}\left(\mathbf{x},\omega\right) \otimes \hat{\mathbf{C}}_{\lambda}^{\dagger}\left(\mathbf{x},\omega\right)\rangle_{\beta} = N\left(\omega\right)+1$. Now, for the case in question here, we first note that in the limit of $T \to 0$, using the commutation relations, the average gives

$$\begin{aligned}&\left\langle \hat{C}_{a}\left(\mathbf{x}_{3},\omega_{3}\right)\hat{C}_{b}\left(\mathbf{x}_{4},\omega_{4}\right)\hat{C}_{m}^{\dagger}\left(\mathbf{x}_{1},\omega_{1}\right)\hat{C}_{n}^{\dagger}\left(\mathbf{x}_{2},\omega_{2}\right)\right\rangle_{\beta\to\infty} = \\ &\qquad \delta_{bm}\delta_{an}\delta\left(\mathbf{x}_{4}-\mathbf{x}_{1}\right)\delta\left(\mathbf{x}_{3}-\mathbf{x}_{2}\right)\delta\left(\omega_{4}-\omega_{1}\right)\delta\left(\omega_{3}-\omega_{2}\right) \\ &\qquad + \delta_{am}\delta_{bn}\delta\left(\mathbf{x}_{3}-\mathbf{x}_{1}\right)\delta\left(\mathbf{x}_{4}-\mathbf{x}_{2}\right)\delta\left(\omega_{3}-\omega_{1}\right)\delta\left(\omega_{4}-\omega_{2}\right).\end{aligned} \tag{III.171}$$

Since there are two combinations possible here, there are two possibilities for the quantum mechanical analogue. For both cases, we now take two separate harmonic oscillators and write either $\langle \hat{b}\hat{a}\hat{b}^{\dagger}\hat{a}^{\dagger}\rangle$ and $\langle \hat{b}\hat{a}\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle$; both of which are equal to $\langle \hat{b}\hat{b}^{\dagger}\hat{a}\hat{a}^{\dagger}\rangle$, so there is actually only one equivalent. Obviously, this wouldn't necessarily be true for other combinations of operators. To work out the thermal average, we proceed analogously to previously by taking a simpler, two non-interacting particle analogy and writing

$$\begin{aligned}\left\langle \hat{b}\hat{b}^{\dagger}\hat{a}\hat{a}^{\dagger}\right\rangle &= \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\hat{b}\hat{b}^{\dagger}\hat{a}\hat{a}^{\dagger}\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]} = \frac{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle m|\otimes\langle n|\, e^{-\beta\hat{H}}\hat{b}\hat{b}^{\dagger}\hat{a}\hat{a}^{\dagger}\,|n\rangle\otimes|m\rangle}{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle m|\otimes\langle n|\, e^{-\beta\hat{H}}\,|n\rangle\otimes|m\rangle} \\ &= \frac{\sum_{n=0}^{\infty}\left(n+1\right)e^{-\beta\hbar\omega_{a}(n+1/2)}\sum_{m=0}^{\infty}\left(m+1\right)e^{-\beta\hbar\omega_{b}(m+1/2)}}{\sum_{n=0}^{\infty}e^{-\beta\hbar\omega_{a}(n+1/2)}\sum_{m=0}^{\infty}e^{-\beta\hbar\omega_{b}(m+1/2)}} \\ &= \left[\frac{1}{e^{\beta\hbar\omega_{a}}-1}+1\right]\times\left[\frac{1}{e^{\beta\hbar\omega_{b}}-1}+1\right] = \left[N\left(\omega_{a}\right)+1\right]\left[N\left(\omega_{b}\right)+1\right]\end{aligned} \tag{III.172}$$

where the Hamiltonian $\hat{H} = \hbar\omega_{a}(\hat{a}^{\dagger}\hat{a}+1/2)+\hbar\omega_{b}(\hat{b}^{\dagger}\hat{b}+1/2)$. From this result, it seems a sensible conclusion that the form our thermal averages must take is

$$\left\langle \hat{C}_a\left(\mathbf{x}_3,\omega_3\right)\hat{C}_b\left(\mathbf{x}_4,\omega_4\right)\hat{C}_m^\dagger\left(\mathbf{x}_1,\omega_1\right)\hat{C}_n^\dagger\left(\mathbf{x}_2,\omega_2\right)\right\rangle_\beta = \left[N\left(\omega_1\right)+1\right]\left[N\left(\omega_2\right)+1\right]\times \\ \Big[\delta_{bm}\delta_{an}\delta\left(\mathbf{x}_4-\mathbf{x}_1\right)\delta\left(\mathbf{x}_3-\mathbf{x}_2\right)\delta\left(\omega_4-\omega_1\right)\delta\left(\omega_3-\omega_2\right) \\ +\delta_{am}\delta\left(\mathbf{x}_3-\mathbf{x}_1\right)\delta\left(\omega_3-\omega_1\right)\delta_{bn}\delta\left(\mathbf{x}_4-\mathbf{x}_2\right)\delta\left(\omega_4-\omega_2\right)\Big]. \tag{III.173}$$

However, we emphasise again, as discussed in [95], that the mathematical formalities behind this relationship are not to be underestimated. Similar expressions hold for other combinations of creation and annihilation operators which allows one to work out the thermal averages of the commutations of polariton operators that arise in (III.161), for example, one finds that

$$\left\langle\left[\hat{C}^b\left(\mathbf{x}_3,\omega_3\right)\hat{C}^j\left(\mathbf{x}_4,\omega_4\right),\hat{C}_m^\dagger\left(\mathbf{x}_1,\omega_1\right)\hat{C}_n^\dagger\left(\mathbf{x}_2,\omega_2\right)\right]\right\rangle = \left[N\left(\omega_1\right)+N\left(\omega_2\right)+1\right]\times \\ \Big[\delta_{jm}\delta_{bn}\delta\left(\mathbf{x}_4-\mathbf{x}_1\right)\delta\left(\mathbf{x}_3-\mathbf{x}_2\right)\delta\left(\omega_4-\omega_1\right)\delta\left(\omega_3-\omega_2\right) \\ +\delta_{bm}\delta_{jn}\delta\left(\mathbf{x}_3-\mathbf{x}_1\right)\delta\left(\mathbf{x}_4-\mathbf{x}_2\right)\delta\left(\omega_3-\omega_1\right)\delta\left(\omega_4-\omega_2\right)\Big]. \tag{III.174}$$

As discussed above, the interaction Hamiltonian (III.119) can be broken up into three parts then the first order correction to the correlation function can be shown to be given by

$$\left\langle\hat{\mathbf{E}}\left(\mathbf{x},t\right)\otimes\hat{\mathbf{E}}\left(\mathbf{x}',t\right)\right\rangle_\beta^{(1)} = -\frac{i}{\hbar}\sum_\lambda\int d^3\mathbf{x}_1\int d^3\mathbf{x}_2\int_0^\infty d\omega_1\int_0^\infty d\omega_2\,\left[N\left(\omega_1\right)+N\left(\omega_2\right)+1\right]\times \\ \left[e^{-i(\omega_1+\omega_2)t}\int_{-\infty}^t dt'\,\mathbf{f}_\mathbf{E}\left(\mathbf{x},\mathbf{x}_1,\omega_1\right)\cdot\boldsymbol{\xi}^\lambda\left(\mathbf{x}_1,\mathbf{x}_2,\omega_1,\omega_2,t'\right)\cdot\mathbf{f}_\mathbf{E}^T\left(\mathbf{x}',\mathbf{x}_2,\omega_2\right)+\text{c.c}\right] \tag{III.175}$$

where the matrix $\boldsymbol{\xi}$ is that of (III.128), representing the rate of excitation of polariton pairs. In this expression, it can be seem that the temperature dependence exists only in the term $[N(\omega_1)+N(\omega_2)+1]$ which goes to 1 as $T\to 0$ giving the perturbation to the vacuum state. Furthermore, note that the time-independent correlation function (III.159) of the fluctuation dissipation theorem includes the term $\coth(\hbar\omega_1/k_BT)$. Our thermal factor $[N(\omega_1)+N(\omega_2)+1]$ in this first order correction takes the same form when $\omega_1=\omega_2$. In some sense, this new factor is a generalisation of the time-independent case for when the frequency is not conserved.

Example: Oscillating Homogeneous Dielectric

We now consider the specific case of a homogeneous medium oscillating back and forth with a velocity given by $\mathbf{v}(\mathbf{x},t)=v(t)\,\hat{\mathbf{z}}=z_0\nu\cos(\nu t)\,\hat{\mathbf{z}}$. In this case, we assume that the only time dependence comes from the motion and set $\delta\alpha\to 0$, and therefore using (III.135), for the electric field correlation function, one then finds that the change to the correlation function (III.159), is found to be

$$\begin{aligned}\left\langle\hat{\mathbf{E}}(\mathbf{x},t)\otimes\hat{\mathbf{E}}(\mathbf{x}',t)\right\rangle_\beta^{(1)} &= -\frac{i\mu_0}{\pi c^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int_0^\infty d\omega_1\int_0^\infty d\omega_2\,\omega_1^2\omega_2^2\Big[N(\omega_1)+N(\omega_2)+1\Big]\times\\ &\Big\{\Phi(\omega_1,\omega_2,t)\sqrt{\mathrm{Im}[\epsilon(\omega_1)]\,\mathrm{Im}[\epsilon(\omega_2)]}\,\mathbf{G}(\mathbf{k},\omega_1)\cdot\boldsymbol{\mathcal{A}}(\mathbf{k},\omega_1,\omega_2)\cdot\mathbf{G}(\mathbf{k},\omega_2)\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\\ &\qquad+\text{c.c}\Big\}\end{aligned} \tag{III.176}$$

where the time dependence is contained in the term

$$\begin{aligned}\Phi(\omega_1,\omega_2,t) &= e^{-i(\omega_1+\omega_2)t}\int_{-\infty}^{t}dt'\,e^{i(\omega_1+\omega_2)t'}v(t')\\ &=\lim_{\eta\to 0}\frac{z_0\nu}{2i}\left[\frac{e^{i\nu t}}{\omega_1+\omega_2+\nu-i\eta}+\frac{e^{-i\nu t}}{\omega_1+\omega_2-\nu-i\eta}\right].\end{aligned} \tag{III.177}$$

The time dependent term Φ has the effect of creating a time varying phase in $\left\langle\hat{\mathbf{E}}(\mathbf{x},t)\otimes\hat{\mathbf{E}}(\mathbf{x}',t)\right\rangle_\beta^{(1)}$. To further understand (III.177), one can write the fractions in terms of poles and principal parts

$$\frac{1}{\omega_1 + \omega_2 \pm \nu - i\eta} = P\frac{1}{\omega_1 + \omega_2 \pm \nu} + \pi i \delta(\omega_1 + \omega_2 \pm \nu) \qquad \text{(III.178)}$$

where P indicates the principal value. Thus a large part of the contribution to first order changes to the two point function arise when the two frequencies ω_1 and ω_2 sum to ν at which point the delta function argument from the second fraction is satisfied. Note that since all three frequencies are positive, the delta function $\delta(\omega_1 + \omega_2 + \nu)$ is never satisfied. The two principal parts also provide a contribution to the correlation for frequencies that are close to matching the condition $\omega_1 + \omega_2 = \pm\nu$ but go to zero as $1/\omega$. A contour representation of the function Φ is plotted in Figure III.4.6.

In section III.4.3, the polariton pair production for such a system was analysed where, to keep things simple, the Lorentzian dielectric function (III.93) was used. To build on these results, here we use the same dielectric function. Here we consider the form of the spectral correlation function $\boldsymbol{\rho}^{\beta}(\mathbf{k}, \omega_1, \omega_2, \mathbf{d}, t)$ defined such that

$$\left\langle \hat{\mathbf{E}}\left(\mathbf{x}, t\right) \otimes \hat{\mathbf{E}}\left(\mathbf{x}', t\right) \right\rangle_{\beta}^{(1)} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_0^{\infty} d\omega_1 \int_0^{\infty} d\omega_2 \, \boldsymbol{\rho}^{\beta}(\mathbf{k}, \omega_1, \omega_2, \mathbf{d}, t). \tag{III.179}$$

The parameter $\mathbf{d}$ is the distance between the positions of the two electric field amplitudes $\mathbf{x} - \mathbf{x}'$. As can be seen in Figures III.4.6(b) and III.4.6(b), the integrand is negligible for the most part but diverges for certain values of frequency and wave–vector. Here, we give as an example the xx component of $\boldsymbol{\rho}^{\beta}$ but the same analysis holds true for the other components. It has already been indicated that it diverges when the sum of the frequencies ω_1 and ω_2 is close to the oscillating frequency of the medium. As can be seen in Figure III.4.6(b) and in Figure III.4.6(b), there is also a large contribution to the correlation function when either one of the frequencies matches the resonant frequency of the dielectric, in this example there is only ω_0. As an example, Figure III.4.6(b) illustrates the integrand when the two frequencies match the condition $\omega_1 + \omega_2 = \nu$; it indicates that there is a significant correlation when the dispersion relation of the electromagnetic field within a dielectric $k^2 - \omega^2\epsilon(\omega) = 0$ is satisfied. This contribution comes from the Green's function terms in (III.176) as can be seen in the time-independent integrand of (III.159) plotted in Figure III.4.6(a). The time-dependence serves to add the additional shifted condition $k^2 - \omega_+^2\epsilon(\omega_+) = 0$ where in this specific case $\omega_+ = \nu - \omega$. The time dependent exponential in (III.177) as well as the position dependent term in (III.176) contribute a varying phase, the spatial part of which is observed in Figure III.4.6(b). These effects are heavily dependent upon the absorption of the medium. Indeed as the absorption is set zero, all perturbation terms vanish and the motion of the dielectric has no effect on the quantum vacuum. This can be seen by the presence of the imaginary parts of the permittivity functions in (III.176).

In this chapter, we introduced the macroscopic QED model that allows for the canonical quantisation of electromagnetism within a dispersive, dissipative magneto-dielectric. Our goal was to use this model to analyse the form of the quantum vacuum in the vicinity of time-dependent dielectrics. However, we also saw that for systems containing even the simplest time-dependencies, it becomes unfeasible to use the same canonical quantisation procedure used in the stationary, time-independent case. Here, we used the canonical perturbation theory to consider the first order corrections to the time-independent wave function and to the two point correlation functions giving us a route to assessing how these time-dependencies change the quantum vacuum.

The canonical approach seems to not be well suited to dealing with time-dependent systems due to these difficulties in the quantisation. Furthermore, this perturbative approach appears to perhaps not be the most suited to calculating higher order terms in macroscopic QED.

We will now take advantage of another set of tools at our disposition in quantum field theory and consider whether the path integral approach to quantum field theory will prove to be more fruitful for assessing the changes to the quantum vacuum brought about by time-dependent dielectrics.

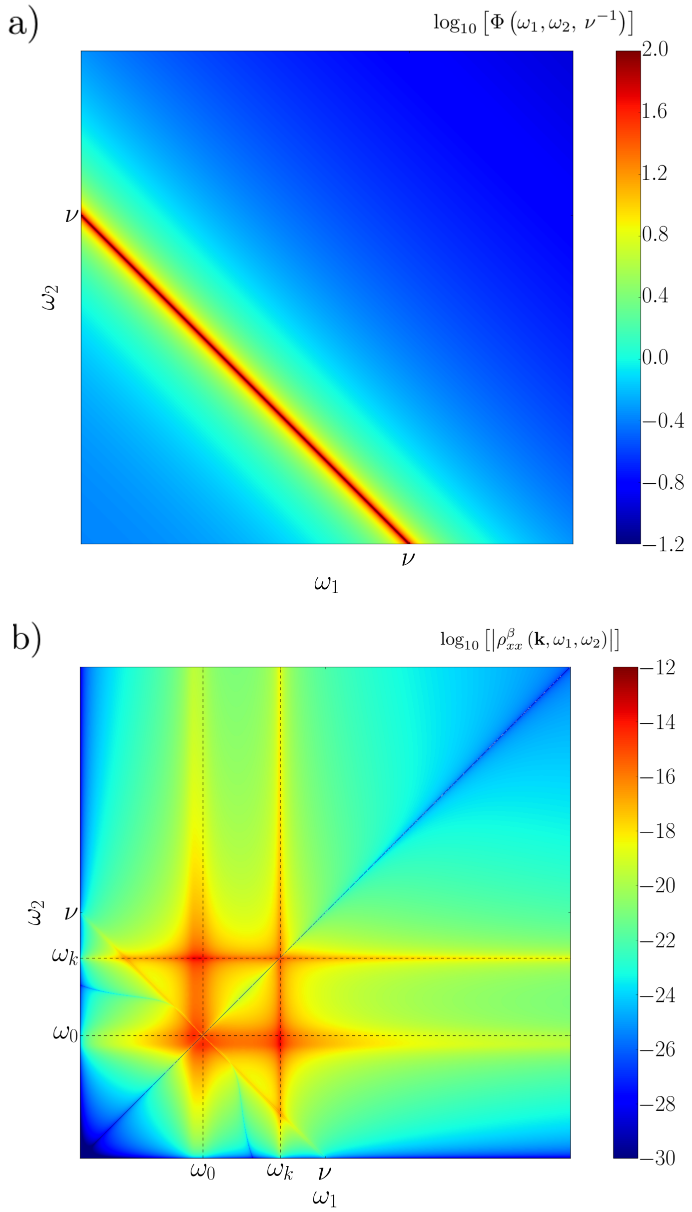

Figure III.12: a) The function Φ ensures that a large part of the contribution to the correlation between the electric field amplitude at two points in space comes from pairs of polaritons whose frequencies ω_1 and ω_2 add up to the frequency of oscillation of the medium ν, i.e $\omega_1 + \omega_2 = \nu$. Again, this is due to conservation of momentum. Here we arbitrarily chose at a fixed time $t = 1/\nu$. b) The logarithm of the absolute value of the xx-component of the spectral function $\log_{10}\left[|\rho^{\beta}_{xx}(\mathbf{k}, \omega_1, \omega_2)|\right]$ for a Lorentzian dielectric whose permittivity is given by (III.93). The (ω_1, ω_2) plane is shown with the arbitrary wave–vector $|\mathbf{k}| = 1.5\,\omega_0/c$ (with the azimuthal angle $\phi = \pi/2$ and polar angle $\theta = \pi/2$) and where we have set the temperature $T = k_B/\hbar\omega_0$ and taken a fixed distance $\mathbf{x} - \mathbf{x}' = \hat{\mathbf{k}}$. There is a large contribution when the delta function in (III.177) is satisfied, i.e when the sum of the polariton pair frequencies add up to the frequency of oscillation of the medium. However, there is also a large contribution to the correlation function when either ω_1 or ω_2 equal the resonant frequency of the material ω_0. These correspond to the horizontal and vertical lines closest to the origin. There are also large contributions when the frequencies satisfying the dispersion relation for light in absorbing medium $k^2 - \omega^2\epsilon(\omega) = 0$. These correspondent to the straight vertical and horizontal lines further away from the origin.

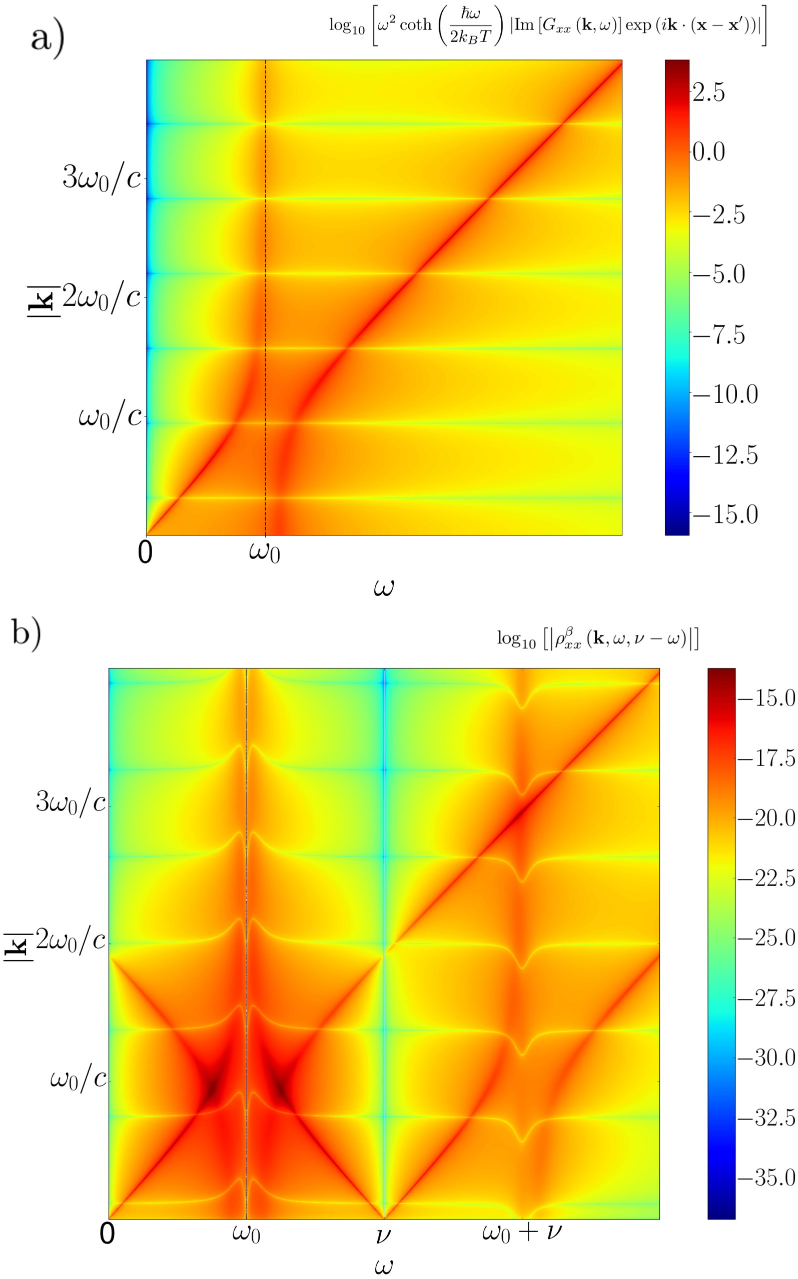

Figure III.13: Figure a) depicts the logarithm of the absolute value of the xx-component of the integrand of (III.159) in the $(\omega, |\mathbf{k}|)$ plane for a Lorentzian dielectric whose permittivity is given by (III.93). This is the correlation when all the time dependencies are set to zero. The majority of the contributions to the correlation function are due to polaritons whose frequency and wave–vector are close to satisfying the dispersion relation $k^2 - \omega^2\epsilon(\omega) = 0$. In Figure b), the logarithm of the absolute value of the xx-component of the spectral function $\log_{10}\left[|\rho^{\beta}_{xx}(\mathbf{k},\omega_1,\omega_2)|\right]$ is plotted in the $(\omega, |\mathbf{k}|)$ plane where we have taken the delta function in (III.177) to be satisfied, i.e $\omega_2 = \nu - \omega_1$. Again, there is a large contribution to the correlation function when the frequencies are close to the resonances of the material or close to satisfying the dispersion relation $k^2 - \omega^2\epsilon(\omega) = 0$ is satisfied. In both plots, the periodic horizontal curves are due to the phase term $\exp[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]$ where we have taken the arbitrary distance $\mathbf{x}-\mathbf{x}' = 5\hat{\mathbf{k}}$ to be fixed.

Chapter IV

The Path Integral Formulation of Macro QED

In the previous chapter, we introduced macroscopic quantum electrodynamics, a relatively new model allowing the full quantisation of absorbing, dispersive magneto-dielectrics interacting with the electromagnetic field. In essence, the main purpose of this model is to be able to perform electrodynamic calculations of large scale (macroscopic) bodies subject to quantum scale electromagnetic fields.

In this thesis, we are primarily concerned with how the non-uniform motion, or general time-dependence of the response functions, of such entities modifies the electromagnetic vacuum. However, time-dependent systems are inherently problematic and even for the simplest of cases, the canonical quantisation approach requires the use of perturbative methods. The complexity of macroscopic QED means that working out higher order pertubative terms using the canonical formalism covered in the previous section is a time consuming process especially when considering the dynamics of time-dependent systems in thermal equilibrium.

In this chapter, we explore an alternative formulation of macroscopic QED afforded by quantum field theory: the path integral. As we will see, the path integral approach is a powerful tool for establishing a perturbation series based on a set of pre-established rules known as the Feynman rules. We begin by giving a brief introduction to the path integral method and showing how it can be applied to Macroscopic QED using the standard methods of quantum field theory. Ultimately, we wish to use this model to compute higher order corrections to the correlation functions between field amplitudes such as those of section III.4.6 in order to understand how the time-dependence of magneto-dielectrics changes the vacuum. However, as we saw in the previous chapter, these are computed by taking the zero temperature limit of the thermal correlators. In order to deal with time dependent thermal systems, we will use a model often used for non-equilibrium systems: the Schwinger-Keldysh path integral. We will briefly introduce this method and provide some preliminary remarks on adapting this method to the model used for Macroscopic QED, leaving the full adaptation for future work.

IV.1 Introduction to Path integrals

Traditionally, resting upon the notions of causality and predetermination, classical physics predicts that individual particles trace out exact paths throughout space as time evolves. The action principal described in section I.2 states that this path will be that which minimises the action. Our current understanding of quantum mechanics seems to disrupt this world view by suggesting that particles no longer follow predetermined, well-defined trajectories despite many attempts to infer otherwise[1].

The concept of the path integral goes a step further by relying on the somewhat counter-intuitive notion that at the scale where quantum mechanics becomes important, the path taken by this particle is no longer unique, and in fact, no longer has the same meaning as the path described by classical physics. The mathematics of the path integral is that of a particle that simultaneously traces out all possible paths between its starting point and its end point, with each path contributing its own probability of being the "actual" path taken. Classical physics, in this view, is seen as being the limit in which only one path contributes over-

[1] See for example the Broglie-Bohm interpretation of quantum mechanics [96].

whelmingly more significantly than all other possible paths.

The path integral has its origin in work by Norbert Wiener [97] and Paul Dirac [98] but it is to Feynman that most of the credit is given for more rigorously developing the method. Several routes led Feynman to this formulation, one of which can be seen by considering the role of time in quantum field theory; in the canonical quantum field theory, and in quantum mechanics in general, time is picked out as being somehow special. On a superficial level, this can be seen in the commutation relations (I.42) which are always defined at equal times, unlike for the spatial coordinates. In the Wheeler-Feynman absorber theory of Section I.4.2, in a universe made entirely of harmonic oscillators, the extension of an oscillator X_n at some moment in time t is invariably written in terms of the extension of all other oscillators in the entire universe at every point in time, past and present (due to the time symmetry of their theory) multiplied by some memory kernel G, i.e

$$\mathsf{X}_n(t) = \sum_m \int_{-\infty}^{\infty} dt' \, \mathsf{G}(t,t')\mathsf{X}_m(t'). \tag{IV.1}$$

where the subscript n/m is the particle identification. If one attempted to quantise this action at a distance theory, one would necessarily require some form of commutation relation between the extensions X_n at *different* times. The path integral skirts around this issue in a natural way by removing the need for any commutators, as we shall now see.

IV.1.1 The Path Integral for Single Particle Quantum Mechanics

The concept of the path integral essentially arose thanks to special properties of the evolution of wave functions. The time evolution of any quantity obeying an equation of motion of the form

$$\hat{H}(\tau)\psi = \frac{\partial}{\partial\tau}\psi \tag{IV.2}$$

can be given as

$$\psi(\tau) = \psi_0 \exp\left(\int\limits_{\tau_0}^{\tau} d\tau' \; H(\tau')\right). \tag{IV.3}$$

In the case of the Schroedinger equation, the wave function evolves according to (IV.2) with the parameter $\tau = -it/\hbar$. One side effect of this simple time evolution is that the amplitude of the wave function at any position and time can be written in terms of previous amplitudes as

$$\psi(\mathbf{x}, t) = \int d^3\mathbf{x}' \; K\left(\mathbf{x}, \mathbf{x}', t, t'\right) \psi(\mathbf{x}', t') \tag{IV.4}$$

where the quantity K is known as the *propagator*. The quantum mechanical propagator $K\left(\mathbf{x}, \mathbf{x}', t, t'\right)$ expresses the transition amplitude of a particle between the space-time points $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$, the absolute square of which gives us the probability of measuring a particle at the final position. The same propagation holds for this 'initial' amplitude $\psi(\mathbf{x}', t')$ allowing us to write this in turn, in terms of the some previous amplitude as

$$\psi(\mathbf{x}', t') = \int d^3\mathbf{x}_0 \; K\left(\mathbf{x}', \mathbf{x}_0, t', t_0\right) \psi(\mathbf{x}_0, t_0). \tag{IV.5}$$

Substituting (IV.5) into (IV.4) reveals the compounding law of propagators

$$K\left(\mathbf{x}, \mathbf{x}_0, t, t_0\right) = \int d^3\mathbf{x}' \; K\left(\mathbf{x}, \mathbf{x}', t, t'\right) K\left(\mathbf{x}', \mathbf{x}_0, t', t_0\right). \tag{IV.6}$$

It is worth noting that this only holds due to the first derivative in time in the equations of motion (IV.2). The analogous expression for a scalar field obeying a wave equation would be more complicated and involve time derivatives of the Green's function as can be seen from Kirchoff's integral theorem [99]. Writing the propagator in the form of (IV.6) is equivalent to writing that the propagator between two space-time points $(\mathbf{x}_0, t_0)$ and $(\mathbf{x}, t)$ is the propagator between the initial point $(\mathbf{x}_0, t_0)$ and some intermediary point $(\mathbf{x}', t')$ multiplied by the propagator between this point and the final point $(\mathbf{x}, t)$ but summed over all possible intermediary positions $\mathbf{x}'$ at time t'. We can carry out the same procedure for N intermediary

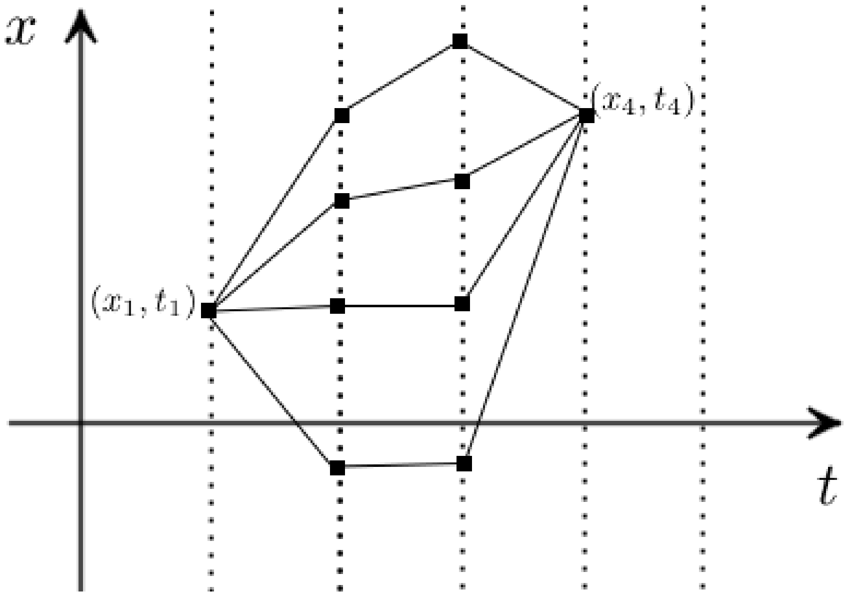

Figure IV.1: A few example paths between spacetime points (x_1, t_1) and (x_4, t_4). In reality, there is an infinite number of paths between these two points. In the path integral, all paths contribute a phase factor to the integral. The classical path is the only one that survives in the classical limit of $\hbar \to 0$.

points leading to

$$K(\mathbf{x}, \mathbf{x}_0, t, t_0) = \prod_{i=1}^{N} \int d^3\mathbf{x}_i \; K(\mathbf{x}, \mathbf{x}_N, t, t_N) K(\mathbf{x}_N, \mathbf{x}_{N-1}, t_N, t_{N-1}) \\ \quad ... \times K(\mathbf{x}_2, \mathbf{x}_1, t_2, t_1) \, K(\mathbf{x}_1, \mathbf{x}_0, t_1, t_0) \, . \tag{IV.7}$$

Figure IV.1.1 represents the possible paths between two spacetime points illustrating (IV.7). Each segment of each path in Figure IV.1.1 has a corresponding weight proportional to a Green's function. Integrating over all of these paths is equal to the direct path.

In bra-ket notation, where the wave function can be written as the inner product $\psi(\mathbf{x}, t) = \langle \mathbf{x}, t | \psi \rangle$, the propagator is written as the inner product

$$\langle \mathbf{x}, t | \mathbf{x}_0, t_0 \rangle = \langle \mathbf{x}, t | \prod_{j=1}^{N} \int d^3\mathbf{x}_j ... | \mathbf{x}_j, t_j \rangle \langle \mathbf{x}_j, t_j | ... | \mathbf{x}_0, t_0 \rangle \tag{IV.8}$$

where including another intermediary position is equivalent to inserting the identity

$$1 = \int d^3\mathbf{x}_l \, | \mathbf{x}_l, t_l \rangle \langle \mathbf{x}_l, t_l | \, . \tag{IV.9}$$

Expression (IV.8) indicates that the propagator can be written as a sum over products of propagators between all combinations of the intermediary positions between space-time points $(\mathbf{x}_0, t_0)$ and $(\mathbf{x}, t)$. In the limit of an infinite number of intermediary steps, the notion of path integral becomes apparent: the propagator is given as the sum over all possible paths between the two end points where each path is weighted by the infinite product of these infinitesimal propagators. The weight turns out to be given by

$$\propto e^{\frac{i}{\hbar}S[\mathbf{x}(t)]} \tag{IV.10}$$

where $S[\mathbf{x}(t)]$ indicates the action along the path $\mathbf{x}(t)$. To show this, we work out the infinitesimal propagator $\langle \mathbf{x}_{j+1}, t_{j+1} | \mathbf{x}_j, t_j \rangle$. From here on, we assume that the Hamiltonian is independent of time. Moving to the Heisenberg picture by writing

$$|\mathbf{x}, t\rangle = e^{-i\hat{H}t/\hbar} |\mathbf{x}\rangle \tag{IV.11}$$

the infinitesimal propagator can be expressed as

$$\langle \mathbf{x}_{j+1}, t_{j+1} | \mathbf{x}_j, t_j \rangle = \langle \mathbf{x}_{j+1} | e^{-i\hat{H}\tau/\hbar} | \mathbf{x}_j \rangle \tag{IV.12}$$

where $i\hbar\tau = t_{j+1} - t_j$. Since, the limit for an infinitesimal time step τ is taken, the exponential may be expanded to first order in τ as

$$\begin{aligned} \langle \mathbf{x}_{j+1}, t_{j+1} | \mathbf{x}_j, t_j \rangle &= \langle \mathbf{x}_{j+1} | \left(1 - \frac{i}{\hbar}\hat{H}\tau\right) | \mathbf{x}_j \rangle \\ &= \langle \mathbf{x}_{j+1} | \mathbf{x}_j \rangle - \frac{i\tau}{\hbar} \langle \mathbf{x}_{j+1} | \hat{H} | \mathbf{x}_j \rangle . \end{aligned} \tag{IV.13}$$

In order to proceed, consider the simplest case of a free particle where the Hamiltonian is given by $H = \frac{1}{2}m\dot{\mathbf{x}}^2 = \mathbf{p}^2/2m$. The first term, which amounts to a delta function $\delta^{(3)}(\mathbf{x}_{j+1} - \mathbf{x}_j)$ is replaced with an integral representation. In order to address the second term, we insert the identity

$$1 = \int d^3\mathbf{p} \; |\mathbf{p}\rangle \langle \mathbf{p}| \tag{IV.14}$$

before and after the Hamiltonian operator. This gives

$$\begin{aligned}
\langle \mathbf{x}_{j+1}, t_{j+1} | \mathbf{x}_j, t_j \rangle &= \int \frac{d^3\mathbf{p}_j}{2\pi\hbar} e^{i\mathbf{p}_j\cdot(\mathbf{x}_{j+1}-\mathbf{x}_j)/\hbar} \\
&\quad - \frac{i\tau}{2m\hbar} \int d^3\mathbf{p}_j \int d^3\mathbf{p}'_j \ \langle \mathbf{x}_{j+1} | \mathbf{p}_j \rangle \langle \mathbf{p}_j | \hat{\mathbf{p}}_j^2 \left| \mathbf{p'}_j \right\rangle \langle \mathbf{p}'_j | \mathbf{x}_j \rangle \\
&= \int \frac{d^3\mathbf{p}_j}{2\pi\hbar} e^{i\mathbf{p}_j\cdot(\mathbf{x}_{j+1}-\mathbf{x}_j)/\hbar} - \frac{i\tau}{\hbar} \int \frac{d^3\mathbf{p}_j}{2\pi\hbar} \left(\frac{\mathbf{p}_j^2}{2m} \right) e^{i\mathbf{p}_j\cdot(\mathbf{x}_{j+1}-\mathbf{x}_j)/\hbar}
\end{aligned} \tag{IV.15}$$

where we have used

$$\langle \mathbf{x}_{j+1} | \mathbf{p}_j \rangle = \frac{1}{\sqrt{2\pi\hbar}} \ e^{i\mathbf{p}_j\cdot\mathbf{x}_{j+1}/\hbar}. \tag{IV.16}$$

This can be rewritten as

$$\begin{aligned}
\langle \mathbf{x}_{j+1}, t_{j+1} | \mathbf{x}_j, t_j \rangle &= \int \frac{d^3\mathbf{p}_j}{2\pi\hbar} e^{i\mathbf{p}\cdot(\mathbf{x}_{j+1}-\mathbf{x}_j)/\hbar} \left(1 - \frac{i\tau}{\hbar} H\left(\mathbf{p}_j\right) \right) \\
&= \int \frac{d^3\mathbf{p}_j}{2\pi\hbar} \exp\left(\frac{i}{\hbar} \left[\mathbf{p}_j \cdot \left(\mathbf{x}_{j+1} - \mathbf{x}_j \right) - \tau H\left(\mathbf{p}_j\right) \right] \right)
\end{aligned} \tag{IV.17}$$

by again using the first order Taylor expansion of an exponential $\exp(x) \approx 1 + x$. We identify the exponent as being the discrete action as previously discussed. Using similar expressions for all the infinitesimal propagators, the full propagator can be written as

$$\langle \mathbf{x}, t | \mathbf{x}_0, t_0 \rangle = \prod_j^N \int d^3\mathbf{x}_j \int \frac{d^3\mathbf{p}_j}{2\pi\hbar} \exp\left(\frac{i}{\hbar} \left[\mathbf{p}_j \cdot \left(\mathbf{x}_{j+1} - \mathbf{x}_j \right) - \tau H\left(\mathbf{p}_j\right) \right] \right). \tag{IV.18}$$

In general, the Hamiltonian will depend on both the position $\mathbf{x}$ as well as the momentum. In the limit where $N \to \infty$, the propagator expanded in this manner is symbolically written as

$$\begin{aligned}
\langle \mathbf{x}, t | \mathbf{x}_0, t_0 \rangle &= \int D^3\mathbf{x} \int \frac{D^3\mathbf{p}}{2\pi\hbar} \exp\left(\frac{i}{\hbar} \int_{t_0}^{t} dt_1 \left[\mathbf{p} \cdot \dot{\mathbf{x}} - H\left(\mathbf{x}, \mathbf{p}\right) \right] \right) \\
&= \int D^3\mathbf{x} \int \frac{D^3\mathbf{p}}{2\pi\hbar} \exp\left(\frac{i}{\hbar} S\left[\mathbf{x}, \mathbf{p}\right] \right).
\end{aligned} \tag{IV.19}$$

The symbolic notation for the path integral

$$\prod_{j}^{N} \int d^3\mathbf{x}_j \to \int D^3\mathbf{x} \tag{IV.20}$$

may at first seem odd, as does the fact one has to make use of this discretisation in order to perform the path integral. However, this expression can be understood by a simple comparison to the simplest concept of the derivative. When learning the concept of a derivative, one is usually introduced by taking the gradient of a function over some finite distance

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x+\eta) - f(x)}{\eta} \tag{IV.21}$$

where in the limit of $\eta \to 0$, we call this gradient, the "derivative" and promote the special notation df/dx. For this operation, we create a list of rules that we must apply that depend upon on the function that we wish to derive. These rules are established by using (IV.21) explicitly and then taking the limit $\eta \to 0$ right at the very end. However, this explicit method is not used for every derivative one wishes to take, instead we turn to the list of rules. In the same way, in order to find explicit results for the path integral, the discrete form (IV.18) is often used, and the limit of the time step $\tau \to 0$ is taken right at the end. However, analogously, a set of rules begin to appear and for many cases, we can apply previous results without explicit use of the discrete form, simply using the shorthand (IV.19).

It should be noted that for a generic system, the functional integral (IV.19) is not necessarily well-defined in the same manner that the usual Riemann integral or the derivative is not necessarily well-defined either for any arbitrary function.

Expression (IV.19) is the most general form of the path integral but in our simple case, and in a large class of other situations, this expression can be further simplified by carrying out the integral over momentum. Starting from the discrete form, expression (IV.18), we carry out the N integrals over the momentum p via

Gaussian integration using the well-known result

$$\int d^3\mathbf{p}\, e^{-\frac{1}{2}\mathbf{p}^T\cdot\mathbf{A}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{p}} = \left(\frac{\pi}{\det(\mathbf{A})}\right)^{3/2} e^{\frac{1}{2}\mathbf{b}^T\cdot\mathbf{A}^{-1}\cdot\mathbf{b}}. \tag{IV.22}$$

Here, we identify $\mathbf{A} = (i\tau/\hbar m)\,\mathbb{1}_3$ and $\mathbf{b} = i/\hbar\,(\mathbf{x}_{j+1} - \mathbf{x}_j)$ and carrying out the integral for every value of j leads to the integral

$$\langle \mathbf{x}, t|\mathbf{x}_0, t_0\rangle = \left(\frac{m}{2\pi i\hbar\tau}\right)^{3N/2} \prod_j^N \int d^3\mathbf{x}_j\ \exp\left(\frac{im}{2\hbar\tau}\left(\mathbf{x}_{j+1} - \mathbf{x}_j\right)^2\right). \tag{IV.23}$$

We identify the exponent as containing the discrete Lagrangian and in the continuum limit $\tau \to 0$, we write this in the symbolic form

$$\langle \mathbf{x}, t|\mathbf{x}_0, t_0\rangle = \int D^3\mathbf{x} \exp\left(\frac{i}{\hbar}\int_{t_0}^{t} dt_1\ L\left[\mathbf{x}, \dot{\mathbf{x}}, t_1\right]\right). \tag{IV.24}$$

For most cases of interest, providing that the integral over the momentum is possible in theory, the starting point will be (IV.24) and this is the form of the path integral usually referred to. In order to show how one would calculate a propagator from this expression by, we write this in the discrete form of (IV.23) and perform the Gaussian integrals one by one. One finds the result

$$\langle \mathbf{x}, t|\mathbf{x}_0, t_0\rangle = \left(\frac{m}{2\pi i\hbar\,(t - t_0)}\right)^{3/2} \exp\left(\frac{im\,(\mathbf{x} - \mathbf{x}_0)^2}{2\hbar\,(t - t_0)}\right). \tag{IV.25}$$

In this simple case, one could of course instead apply the step (IV.9) to the full propagator $\langle \mathbf{x}, t|\mathbf{x}_0, t_0\rangle$ leaving us only one integral over momentum to perform which would give us our final result. However, in general (IV.19) is the correct starting point for the path integral approach.

The propagator for the free particle (IV.25) turns out to simply be equivalent to a phase term that depends on the classical action, i.e. $\exp\left(i/\hbar\, S_{cl}\left[\mathbf{x}\right]\right)$. This is in fact the case for any Lagrangian that is quadratic in $\mathbf{x}$ and $\dot{\mathbf{x}}$. This will will also hold the Lagrangian for macroscopic QED as we shall discuss later. At the scale where the action S is of the same order of magnitude as $\hbar$, i.e. at the quantum scale, any possible trajectory *may* in theory contribute to the phase acquired by the particle while travelling between points $\mathbf{x}_0$ and $\mathbf{x}$ (although, as mentioned, for the particular case of quadratic Lagrangians only the classical path ends up counting even at quantum scales). In the classical limit, where the action becomes very large in comparison to $\hbar$, away from the classical path, the phase of the each path changes dramatically between neighbouring paths and their overall contributions average to 0. It is only in the vicinity of the classical path where by definition $\delta S = 0$, that any contribution survives. Knowledge of this allows us to treat with increased ease, the semi-classical limit where the particle follows a path close to the classical path, i.e $\mathbf{x}(t) = \mathbf{x}_{cl} + \delta\mathbf{x}(t)$ where $\mathbf{x}_{cl}$ is the classical path and $\delta\mathbf{x}(t)$ is some small deviation from it. In this case, we expand the action around the classical path

$$\begin{aligned} S\left[\mathbf{x}(t)\right] = S\left[\mathbf{x}_{cl}\right] + \int dt\, \left.\frac{\delta L}{\delta \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{cl}} \cdot \delta\mathbf{x}(t) \\ + \int dt\ \mathrm{Tr}\left[\left.\frac{\delta^2 L}{\delta\mathbf{x}\otimes\delta\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{cl}} \cdot \left[\delta\mathbf{x}(t)\otimes\delta\mathbf{x}(t)\right]\right] + \mathcal{O}(S^{(3)}) \end{aligned} \tag{IV.26}$$

where by definition, the first order disappears when expanding around the classical path. If the action is at most quadratic in the canonical variables, no orders higher than the second survive and can be discarded. Using $D^3\mathbf{x} \equiv D^3\left(\delta\mathbf{x}\right)$, the path integral can be written to second order as

$$\begin{aligned} \langle \mathbf{x}, t|\mathbf{x}_0, t_0\rangle &= \int D^3\mathbf{x} \exp\left(\frac{i}{\hbar} S\left[\mathbf{x}\right]\right) \\ &= \exp\left(\frac{i}{\hbar} S\left[\mathbf{x}_{cl}\right]\right) \int D^3\left(\delta\mathbf{x}\right) \exp\left(\frac{i}{\hbar} S^{(2)}\left[\delta\mathbf{x}\right]\right) \end{aligned} \tag{IV.27}$$

where the second order term is found by

$$S^{(2)}\left[\delta\mathbf{x}\right] = \int dt\ \mathrm{Tr}\left[\left.\frac{\delta^2 L}{\delta\mathbf{x}\otimes\delta\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{cl}} \cdot \left(\delta\mathbf{x}(t)\otimes\delta\mathbf{x}(t)\right)\right]. \tag{IV.28}$$

In the simple case of the free particle, this second order action is given by $S^{(2)}\left[\delta\mathbf{x}\right] = \int dt\ \frac{1}{2}m\left(\delta\dot{\mathbf{x}}\right)^2$. Using the same methods as previously shows that this term corresponds to the time dependent factor of (IV.25) while $\exp\left(\frac{i}{\hbar}S\left[\mathbf{x}_{cl}\right]\right)$ corresponds to the exponential term. The advantage of this method is that establishing quantum mechanical results can often be boiled down to finding this second order change to the action, as knowledge of the classical action allows to simply write down the first part of (IV.27).

This concludes our introduction to the path integral formalism for single particle quantum mechanics, but how does this extend to quantum field theory?

IV.1.2 Quantum Field Theory and Macroscopic QED

In quantum field theory, the situation is very similar. The wave function evolves according to

$$\hat{H}(t)\left|\psi\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle \tag{IV.29}$$

where the analogous expression for (IV.11) is given by

$$\left|\psi\left(\mathbf{x},t\right)\right\rangle = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} dt\ \hat{H}(t)\right)\left|\psi\left(\mathbf{x}\right)\right\rangle. \tag{IV.30}$$

As such, the propagator of the states of the system can be expressed in an analogous manner via a path integral. However, in this case, the path integral is not over all possible paths of some particle but over all the possible *configurations of the fields*. For example, for a scalar field whose action is given as some functional of the canonical variables $S\left[\phi,\Pi_\phi\right]$, in general, the propagator of the wavefunction is given by the path integral

$$\int D\phi\int D\Pi_\phi\ \exp\left\{\frac{i}{\hbar}\left(\Pi_\phi\cdot\dot{\phi}-H\left[\phi,\Pi_\phi\right]\right)\right\}. \tag{IV.31}$$

In all the cases considered in this thesis, the Hamiltonian H is quadratic in the fields and as such, in analogy to quantum mechanics, we instead begin from the more concise expression

$$\int D\phi\ \exp\left(\frac{i}{\hbar}S\left[\phi\right]\right). \tag{IV.32}$$

For the time-dependent Hamiltonian of Section III.4, this fact is somewhat more subtle and we shall give a proof of this in Section IV.3.2.

IV.2 Partition Function and Generating Functional

A useful notion within path integral methods used extensively to calculate correlation functions is the concept of the generating functional. Macroscopic QED is composed of an infinite number of harmonic oscillators; therefore before considering the generating functional of macroscopic QED, let us first consider the simplest case of the 1-dimensional harmonic oscillator.

IV.2.1 Generating Functional for Single Particle Quantum Mechanics

Here we will consider the simplest case of the 1-dimensional harmonic oscillator of displacement X from its equilibrium position at some finite temperature. The Lagrangian, canonical momentum and Hamiltonian for this system are again given by expressions (I.26)-(I.28).

The motivation behind the generating functional is that often we require the explicit calculation of thermal expectation values such as $\left\langle \hat{\mathsf{X}}^2 \right\rangle_\beta$ for example, where $\beta = 1/k_B T$. As usual, this can be written as

$$\left\langle \hat{\mathsf{X}}^2 \right\rangle_\beta = \frac{\int d\mathsf{X} \left\langle \mathsf{X} \right| \mathsf{X}^2 e^{-\beta \hat{H}} \left| \mathsf{X} \right\rangle}{\int d\mathsf{X} \left\langle \mathsf{X} \right| e^{-\beta \hat{H}} \left| \mathsf{X} \right\rangle}. \tag{IV.33}$$

Consider the exponential in the denominator; superficially this appears to just be $\exp(i\hat{H}t/\hbar)$ with $t \to i\hbar\beta$. It turns out that if we proceed in a similar fashion as for the derivation of (IV.24), but replace the small real time steps by small steps in imaginary time, i.e $t \to i\tau$, with periodic boundary conditions, i.e where the end points satisfy

$$\mathsf{X}(0) = \mathsf{X}(\hbar\beta) \tag{IV.34}$$

and integrate over all possible starting positions for *closed* paths (i.e starting and ending at the same point), we can compute the denominator using only path integrals without having to resort to the usual operator methods. Applying the path integral methods, this can be written as

$$
\begin{aligned}
\int d\mathsf{X} \langle \mathsf{X}| e^{-\beta \hat{H}} |\mathsf{X}\rangle &= \int D\mathsf{X}(\tau)\ \exp\left[-\frac{1}{\hbar}\int\limits_0^{\hbar\beta} d\tau\ L(\mathsf{X}, \partial_\tau \mathsf{X})\right] \\
&= \int D\mathsf{X}(\tau)\ \exp\left\{\frac{1}{2}\int\limits_0^{\beta} d\tau \left[m\left(\frac{\partial \mathsf{X}}{\partial \tau}\right)^2 + \kappa \mathsf{X}^2\right]\right\}.
\end{aligned}
\tag{IV.35}
$$

We can integrate by parts and re-express this as

$$
\int d\mathsf{X} \langle \mathsf{X}| e^{-\beta \hat{H}} |\mathsf{X}\rangle = \int D\mathsf{X}(\tau)\ \exp\left\{\left[\mathsf{X}\frac{\partial \mathsf{X}}{\partial \tau}\right]_0^\beta + \frac{1}{2}\int\limits_0^{\beta} d\tau\ \left(-m\mathsf{X}\frac{\partial^2 \mathsf{X}}{\partial \tau^2} + \kappa \mathsf{X}^2\right)\right\}
\tag{IV.36}
$$

where using (IV.34), this is just equivalent to

$$
\int d\mathsf{X} \langle \mathsf{X}| e^{-\beta \hat{H}} |\mathsf{X}\rangle = \int D\mathsf{X}(\tau)\ \exp\left[-\frac{1}{2}\int\limits_0^{\beta} d\tau \int\limits_0^{\beta} d\tau'\ \mathsf{X}(\tau) A(\tau,\tau') \mathsf{X}(\tau')\right]
\tag{IV.37}
$$

where we define the operator

$$
A(\tau,\tau') = \left(m\frac{\partial^2}{\partial \tau^2} - \kappa\right)\delta(\tau - \tau').
\tag{IV.38}
$$

Now, this expression is actually the partition function Z. In a similar manner, one may recover the numerator of (IV.33) by defining the *generating functional* by adding a current term to the exponential, i.e.

$$
Z[j,\beta] = \int D\mathsf{X}(\tau)\ \exp\left[-\frac{1}{2}\int\limits_0^{\beta} d\tau \int\limits_0^{\beta} d\tau'\ \mathsf{X}(\tau) A(\tau,\tau') \mathsf{X}(\tau') + \int\limits_0^{\beta} d\tau j(\tau) \mathsf{X}(\tau)\right]
\tag{IV.39}
$$

then by considering the second derivative with respect to the current j and setting the current to zero,

$$\int d\mathsf{X} \langle\mathsf{X}| \mathsf{X}^2 e^{-\beta\hat{H}} |\mathsf{X}\rangle = \left.\frac{\partial^2 Z[j,\beta]}{\partial j(\tau)\partial j(\tau')}\right|_{j=0}$$
$$= \int D\mathsf{X}(\tau)\ \mathsf{X}(\tau)\mathsf{X}(\tau')\exp\left[-\frac{1}{2}\int_0^\beta d\tau \int_0^\beta d\tau'\ \mathsf{X}(\tau)A(\tau,\tau')\mathsf{X}(\tau')\right]. \tag{IV.40}$$

we identify the denominator of (IV.33), i.e. the partition function, to be $Z[0,\beta]$. This allows us to write

$$\left\langle\hat{\mathsf{X}}(\tau)\hat{\mathsf{X}}(\tau')\right\rangle_\beta = \frac{1}{Z[0,\beta]}\cdot\left.\frac{\partial^2 Z[j,\beta]}{\partial j(\tau)\partial j(\tau')}\right|_{j=0}. \tag{IV.41}$$

Given the periodicity in time enforced by (IV.34), we expand the position and current via the discrete sums

$$\begin{aligned}\mathsf{X}(\tau) &= \frac{1}{\beta}\sum_n \tilde{\mathsf{X}}_n e^{i\omega_n\tau}\\ j(\tau) &= \frac{1}{\beta}\sum_n \tilde{j}_n e^{i\omega_n\tau}\end{aligned} \tag{IV.42}$$

where the frequencies $\omega_n = 2\pi n/\beta$ are known as the *Matsubara frequencies*. This allows us to write the exponent in the generating functional as

$$\begin{aligned}I &= -\frac{1}{2}\int_0^\beta d\tau\int_0^\beta d\tau'\ \mathsf{X}(\tau)A(\tau,\tau')\mathsf{X}(\tau') + \int_0^\beta d\tau j(\tau)\mathsf{X}(\tau)\\ &= \frac{1}{2\beta^2}\sum_{n,m}\int_0^\beta d\tau\ \left[m\left(-\omega_n\omega_m - \omega^2\right)\tilde{\mathsf{X}}_n\tilde{\mathsf{X}}_m + 2\,\tilde{j}_n\tilde{\mathsf{X}}_m\right]\ e^{i(\omega_n+\omega_m)\tau}\end{aligned} \tag{IV.43}$$

where $\omega = \sqrt{\kappa/\omega}$. Making use of the integral representation of the Kronecker delta

$$\int_0^\beta d\tau\, e^{i(\omega_n-\omega_m)\tau} = \beta\delta_{nm} \tag{IV.44}$$

and the reality condition $\tilde{\mathsf{X}}_{-n} = \tilde{\mathsf{X}}_n^\star$, the exponent I can be written as

$$\begin{aligned} I &= \frac{1}{2\beta}\sum_n \left[m\left(\omega_n^2-\omega^2\right)\tilde{\mathsf{X}}_n\tilde{\mathsf{X}}_{-n} + 2\,\tilde{j}_n\tilde{\mathsf{X}}_{-n}\right] \\ &= \frac{1}{2\beta}\left[m\left(\omega_0^2-\omega^2\right)\tilde{\mathsf{X}}_0^2 + 2\,\tilde{j}_0\tilde{\mathsf{X}}_0\right] + \frac{1}{\beta}\sum_{n=1}^{\infty}\left[m\left(\omega_n^2-\omega^2\right)\tilde{\mathsf{X}}_n\tilde{\mathsf{X}}_n^\star + \tilde{j}_n\tilde{\mathsf{X}}_n^\star + \tilde{j}_n^\star\tilde{\mathsf{X}}_n\right]. \end{aligned} \tag{IV.45}$$

Because the Fourier components are complex, in order to perform the path integral, we must write these in terms of their real and imaginary parts

$$\begin{aligned} \tilde{\mathsf{X}}_n &= \tilde{\mathsf{X}}_{nR} + i\tilde{\mathsf{X}}_{nI} \\ \tilde{j}_n &= \tilde{j}_{nR} + i\tilde{j}_{nI} \end{aligned} \tag{IV.46}$$

leading to

$$\begin{aligned} I &= \frac{1}{2\beta}\left[m\left(\omega_0^2-\omega^2\right)\tilde{\mathsf{X}}_0^2 + 2\,\tilde{j}_0\tilde{\mathsf{X}}_0\right] \\ &+ \frac{1}{\beta}\sum_{n=1}^{\infty}\left[m\left(\omega_n^2-\omega^2\right)\tilde{\mathsf{X}}_{nR}^2 + 2\,\tilde{j}_{nR}\tilde{\mathsf{X}}_{nR} + m\left(\omega_n^2-\omega^2\right)\tilde{\mathsf{X}}_{nI}^2 + 2\,\tilde{\mathsf{X}}_{nI}\tilde{j}_{nI}\right]. \end{aligned} \tag{IV.47}$$

The partition function can then be calculated by calculating the path integral over the real and imaginary components of the position for every value of n,

$$\begin{aligned} \langle\mathsf{X}|\,e^{-\beta\hat{H}}\,|\mathsf{X}\rangle = &\int D\tilde{\mathsf{X}}_0\;\exp\left\{\frac{1}{2\beta}\left[m\left(\omega_0^2-\omega^2\right)\tilde{\mathsf{X}}_0^2 + 2\,\tilde{j}_0\tilde{\mathsf{X}}_0\right]\right\} \\ &\times\int D\tilde{\mathsf{X}}_{nI}\;\exp\left\{\frac{1}{\beta}\sum_{n=1}^{\infty}\left[m\left(\omega_n^2-\omega^2\right)\tilde{\mathsf{X}}_{nI}^2 + 2\,\tilde{\mathsf{X}}_{nI}\tilde{j}_{nI}\right]\right\} \\ &\times\int D\tilde{\mathsf{X}}_{nR}\;\exp\left\{\frac{1}{\beta}\sum_{n=1}^{\infty}\left[m\left(\omega_n^2-\omega^2\right)\tilde{\mathsf{X}}_{nR}^2 + 2\,\tilde{j}_{nR}\tilde{\mathsf{X}}_{nR}\right]\right\}. \end{aligned} \tag{IV.48}$$

Performing each of the three path integrals leads to the expression for the partition function

$$\begin{aligned} &\langle\mathsf{X}|\,e^{-\beta\hat{H}}\,|\mathsf{X}\rangle \\ &\propto \exp\left\{\frac{1}{2m\beta}\left[\tilde{j}_0\,G\left(i\omega_0\right)\,\tilde{j}_0 + 2\sum_{n=1}^{\infty}\left[\tilde{j}_{nI}\,G\left(i\omega_n\right)\,\tilde{j}_{nI} + \tilde{j}_{nR}\,G\left(i\omega_n\right)\,\tilde{j}_{nR}\right]\right]\right\} \end{aligned} \tag{IV.49}$$

where $G_\omega(\mathrm{i}\omega_n) = 1/(\omega^2 + \omega_n^2)$ is the imaginary frequency Green's function. Taking the appropriate derivatives with respect to the current terms

$$\left\langle \hat{\mathsf{X}}^2 \right\rangle_\beta = \frac{1}{Z[0,\beta]} \cdot \left[\frac{\partial^2 Z[j,\beta]}{\partial j_0 \partial j_0}\bigg|_{j_0=0} + \sum_{n=1}^{\infty} \frac{\partial^2 Z[j,\beta]}{\partial j_n \partial j_n}\bigg|_{j_n=0} \right] \tag{IV.50}$$

allows us to express the correlation function as a sum over the Green's function components

$$\left\langle \hat{\mathsf{X}}^2 \right\rangle_\beta = \left\langle \hat{\mathsf{X}}(0)\hat{\mathsf{X}}(0) \right\rangle_\beta = \frac{1}{m\beta} \left[G_\omega(i\Omega_0) + 2\sum_{n=1}^{\infty} G_\omega(i\Omega_n) \right]. \tag{IV.51}$$

By noting that the poles of the hyperbolic cotangent match the Matsubara fre-

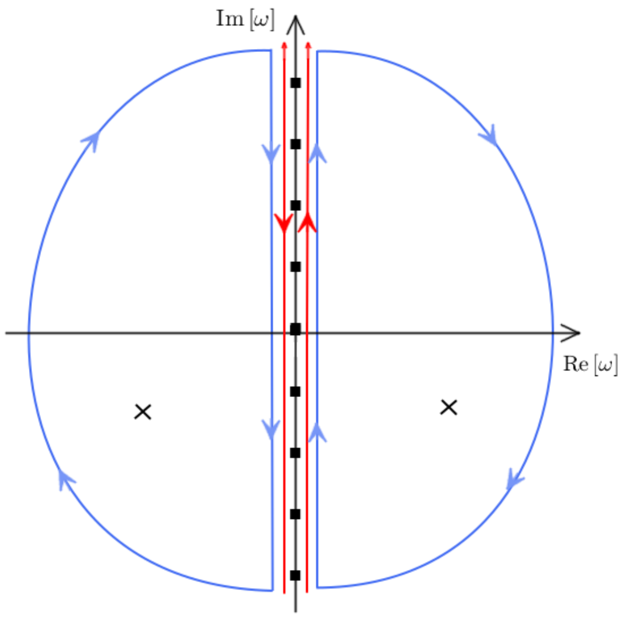

Figure IV.2: The black squares represent the Matsubara frequencies $i\omega_n$. The sum over the Green's function at the Matsubara Frequencies can be written as the red contour where the hyperbolic cotangent is introduced since its residues pick out the Green's function evaluated at those frequencies. This contour can be deformed to become the blue contour the integral over which gives the Green's function evaluated at its poles represented by the black crosses.

quencies, and using the fact that the Green's function is real for imaginary frequencies, this sum can be written as a sum from $n = -\infty$ to $n = \infty$ and then written as a contour integral over the red contour of Figure IV.2 given by

$$\left\langle \hat{\mathsf{X}}^2 \right\rangle_\beta = \oint \frac{d\Omega}{2\pi i} \frac{\hbar}{2m} \coth\left(\frac{\hbar\beta\Omega}{2}\right) G_\omega(\Omega). \tag{IV.52}$$

We can see that this is true by applying the residue theorem[2] to (IV.52) and noting that at each pole of coth $(\hbar\beta\Omega/2)$, the residue is proportional to the Green's function at the pole. Deforming the contour as in Figure IV.2 and evaluating picks up the integrand evaluated at the poles of the Green's function,

$$\left\langle \hat{\mathsf{X}}^2 \right\rangle_\beta = \frac{\hbar}{2m\omega} \coth\left(\frac{\hbar\omega}{2k_B T}\right). \tag{IV.53}$$

The reader will note the analogy to (III.159); in fact, later on once we have shown how to extend this idea to macroscopic QED, we will re-derive (III.159) analogously to (IV.53). Note that if we take the limit as $T \to 0$, we recover the vacuum expectation

$$\left\langle \hat{\mathsf{X}}^2 \right\rangle_\infty = \frac{\hbar}{2m\omega}. \tag{IV.54}$$

For the sake of completeness, we can compare this path integral method to the operator formalism by using the position operator in imaginary time Heisenberg picture where the imaginary time dependence of the operators is found using

$$\frac{\partial \hat{\mathsf{X}}(\tau)}{\partial \tau} = -\frac{1}{\hbar}\left[\hat{H}, \hat{\mathsf{X}}(\tau)\right] \tag{IV.55}$$

from which one can find that the displacement operator can be written in terms of the creation and annihilation operators as

$$\hat{\mathsf{X}}(\tau) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{b}e^{-\omega\tau} + \hat{b}^\dagger e^{\omega\tau}\right) \tag{IV.56}$$

and writing the expectation of $\hat{\mathsf{X}}(\tau)\hat{\mathsf{X}}(\tau')$,

$$\left\langle \hat{\mathsf{X}}(\tau)\hat{\mathsf{X}}(\tau') \right\rangle_\beta = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta\hat{H}}\hat{\mathsf{X}}(\tau)\hat{\mathsf{X}}(\tau')\right] \tag{IV.57}$$

[2]See for example [48]

where $Z = \text{Tr}\left[\exp(-\beta\hat{H})\right]$. Substituting in the expressions for the operators, we can write this as

$$\left\langle \hat{\mathsf{X}}(\tau)\hat{\mathsf{X}}(\tau')\right\rangle_\beta = \left[\sum_{n=0}^{\infty} \langle n| e^{-\beta\hat{H}} |n\rangle\right]^{-1} \sum_{n=0}^{\infty} \langle n| e^{-\beta\hat{H}}\hat{\mathsf{X}}(\tau)\hat{\mathsf{X}}(\tau') |n\rangle . \tag{IV.58}$$

Using the expansion of the Hamiltonian operator in terms of the creation and annihilation operators $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$ and the usual results that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, we can write this as

$$\left\langle \hat{\mathsf{X}}(0)\hat{\mathsf{X}}(0)\right\rangle_\beta = \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}\right]^{-1} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}\left(n+\frac{1}{2}\right). \tag{IV.59}$$

Using the geometric sum result

$$\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{1}{1-e^{-\beta\hbar\omega}} \tag{IV.60}$$

and therefore by extension

$$\begin{aligned} -\frac{1}{\hbar\beta}\frac{d}{d\omega}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)} &= \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}(n+1/2) \\ &= \frac{1}{2}\frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} + \frac{e^{-3\beta\hbar\omega/2}}{\left(1-e^{-\beta\hbar\omega}\right)^2}. \end{aligned} \tag{IV.61}$$

Substituting these into (IV.59) and simplifying again leads to the equal time thermal correlation function

$$\left\langle \hat{\mathsf{X}}^2\right\rangle_\beta = \frac{\hbar}{2m\omega}\coth\left(\frac{\hbar\omega}{2k_BT}\right). \tag{IV.62}$$

as before. The expression (IV.41) generalises to n-th powers of X by taking multiple derivatives, i.e

$$\left\langle \hat{\mathsf{X}}^n\right\rangle_\beta = \frac{1}{Z[0,\beta]}\cdot\left.\frac{\partial^n Z[j,\beta]}{\partial j(0)....\partial j(0}\right|_{j=0}. \tag{IV.63}$$

This concludes our short introduction to the concept of the generating functional of which taking multiple functional derivatives allows one to retrieve the thermal correlation functions of the amplitudes. We will now show how this same concept extends to macroscopic QED and allows us to calculate the thermal correlation functions of stationary, time-independent dielectrics such as (III.159).

IV.2.2 Thermal Macroscopic QED

Our purpose here is to calculate thermal expectations values of the field amplitudes using the path integral methods applied to macroscopic QED. The vacuum correlators we ultimately seek are found by taking the zero temperature limit of these functions. Before addressing the more intricate case of time-dependent media, we must first show that the simpler time-independent cases can be recovered. As far as the present author is aware, this is a novel result, and the path integral formulation has not yet been applied to this model for macroscopic QED.

In order to calculate these correlation functions, we must adapt the concept of the generating functional to macroscopic QED. This is known as thermal quantum field theory.

We can again define a generating functional by simply replacing time by an imaginary time variable periodic in the thermal β. For macroscopic QED, the Lagrangian density is that of [84]. For gauge theories, there is a subtlety in the path integral in that, the gauge invariance of the fields means that integrating over all configurations of the fields results in "double counting" of the integrand leading to nonsensical results, therefore the gauge must be fixed from the start. Here we work in the Coulomb gauge, where $\nabla \cdot \mathbf{A} = 0$ and $\phi = 0$ is no longer a dynamical variable. The generating functional is found by immediately applying the action of [84],

$$Z\left[\mathbf{j}, \mathbf{z}_\omega, \mathbf{w}_\omega\right] = \int D\mathbf{A} \int D\mathbf{X}_\omega \int D\mathbf{Y}_\omega \, \exp\left\{\frac{i}{\hbar} S\left[\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega\right]\right\}$$
$$\times \exp\left\{-\frac{1}{\hbar}\int d^3\mathbf{x} \int_0^{\hbar\beta} d\tau \left[\mathbf{A}\cdot\mathbf{j} + \int_0^\infty d\omega \, \left(\mathbf{X}_\omega \cdot \mathbf{z}_\omega + \mathbf{Y}_\omega \cdot \mathbf{w}_\omega\right)\right]\right\} \tag{IV.64}$$

where we have chosen $\mathbf{j}, \mathbf{z}_\omega$ and $\mathbf{w}_\omega$ to represent the current analogously to

(IV.39). Using the boundary condition on the fields $\mathbf{A}(\mathbf{x},0) = \mathbf{A}(\mathbf{x},\hbar\beta)$, the Euclidean action is defined by

$$S\left[\mathbf{A},\mathbf{X}_\omega,\mathbf{Y}_\omega\right] = i\int d^3\mathbf{x}\int_0^{\hbar\beta} d\tau\ \mathscr{L}_E\left(\mathbf{A},\mathbf{X}_\omega,\mathbf{Y}_\omega\right). \tag{IV.65}$$

In this gauge, the Lagrangian density for a stationary magneto-dielectric medium coupled to the electromagnetic field can be written in terms of the vector potential $\mathbf{A}$ (again, the scalar potential ϕ is zero for all times) and is given as

$$\begin{aligned}\mathscr{L}_0 = {} & \frac{\epsilon_0}{2}\left[\dot{\mathbf{A}}^2 - c^2\left(\nabla\times\mathbf{A}\right)^2\right] - \dot{\mathbf{A}}\cdot\int_0^\infty d\omega\,\alpha\left(\mathbf{x},\omega\right)\mathbf{X}_\omega \\ & + \left(\nabla\times\mathbf{A}\right)\cdot\int_0^\infty d\omega\,\beta\left(\mathbf{x},\omega\right)\mathbf{Y}_\omega + \frac{1}{2}\int_0^\infty d\omega\left[\dot{\mathbf{X}}_\omega^2 - \omega^2\mathbf{X}_\omega^2\right] + \frac{1}{2}\int_0^\infty d\omega\left[\dot{\mathbf{Y}}_\omega^2 - \omega^2\mathbf{Y}_\omega^2\right].\end{aligned} \tag{IV.66}$$

The Euclidean Lagrangian $(t\to i\tau)$ is thus defined as

$$\begin{aligned}\mathscr{L}_E = {} & \frac{\epsilon_0}{2}\left[-\left(\frac{\partial}{\partial\tau}\mathbf{A}\right)^2 - c^2\left(\nabla\times\mathbf{A}\right)^2\right] + i\frac{\partial\mathbf{A}}{\partial\tau}\cdot\int_0^\infty d\omega\,\alpha\left(\mathbf{x},\omega\right)\mathbf{X}_\omega \\ & + \left(\nabla\times\mathbf{A}\right)\cdot\int_0^\infty d\omega\,\beta\left(\mathbf{x},\omega\right)\mathbf{Y}_\omega - \frac{1}{2}\int_0^\infty d\omega\left[\left(\frac{\partial}{\partial\tau}\mathbf{X}_\omega\right)^2 + \omega^2\mathbf{X}_\omega^2\right] \\ & - \frac{1}{2}\int_0^\infty d\omega\left[\left(\frac{\partial}{\partial\tau}\mathbf{Y}_\omega\right)^2 + \omega^2\mathbf{Y}_\omega^2\right].\end{aligned} \tag{IV.67}$$

The generating functional contains exponents that depend on combinations of the electromagnetic potential and either the electric oscillator bath $\mathbf{X}_\omega$ or the magnetic oscillator bath $\mathbf{Y}_\omega$ but never both simultaneously, using the properties of exponentials, the path integral can be written in a more compartmentalised fashion via

$$Z\left[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega\right]=$$
$$\int D\mathbf{A}\,\exp\left(\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau\left\{\epsilon_0\left[\left(\frac{\partial}{\partial\tau}\mathbf{A}\right)^2+c^2\left(\nabla\times\mathbf{A}\right)^2\right]-2\mathbf{A}\cdot\mathbf{j}\right\}\right)$$
$$\times Z\left[\mathbf{A},\mathbf{z}_\omega\right]Z\left[\mathbf{A},\mathbf{w}_\omega\right] \tag{IV.68}$$

where the intermediate path integral over the electric oscillator bath is

$$Z\left[\mathbf{A},\mathbf{z}_\omega\right]=\int D\mathbf{X}_\omega\exp\left(\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\infty d\omega\left\{\left(\frac{\partial}{\partial\tau}\mathbf{X}_\omega\right)^2+\omega^2\mathbf{X}_\omega^2\right.\right.$$
$$\left.\left.-2\left[\mathbf{z}_\omega-i\alpha\left(\mathbf{x},\omega\right)\frac{\partial\mathbf{A}}{\partial\tau}\right]\cdot\mathbf{X}_\omega\right\}\right) \tag{IV.69}$$

and over the magnetic field oscillator bath is

$$Z\left[\mathbf{A},\mathbf{w}_\omega\right]=\int D\mathbf{Y}_\omega\exp\left(\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\infty d\omega\left\{\left(\frac{\partial}{\partial\tau}\mathbf{Y}_\omega\right)^2+\omega^2\mathbf{Y}_\omega^2\right.\right.$$
$$\left.\left.-2\left[\mathbf{w}_\omega+\beta\left(\mathbf{x},\omega\right)\left(\nabla\times\mathbf{A}\right)\right]\cdot\mathbf{Y}_\omega\right\}\right) \tag{IV.70}$$

Note that the terms $Z\left[\mathbf{A},\mathbf{z}_\omega\right]$ and $Z\left[\mathbf{A},\mathbf{w}_\omega\right]$ are functionals of the electromagnetic potential field amplitude $\mathbf{A}$ and as such are included in the integrand of (IV.68). In all three path integrals, the integrals over space and time in the exponents can be integrated by parts as appropriate using the boundary periodic boundary conditions in time and the usual quantum field theory assumption that the fields disappear as $\mathbf{x} \to \pm\infty$. One finds that

$$Z\left[\mathbf{j}, \mathbf{z}_\omega, \mathbf{w}_\omega\right] = \int D\mathbf{A}\,\exp\left(-\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau\,\left\{\epsilon_0\left[\mathbf{A}\cdot\left(\frac{\partial^2}{\partial\tau^2}\mathbf{A}\right) + c^2\mathbf{A}\cdot\nabla\times\left(\nabla\times\mathbf{A}\right)\right] + 2\mathbf{A}\cdot\mathbf{j}\right\}\right) Z\left[\mathbf{A}, \mathbf{z}_\omega\right] Z\left[\mathbf{A}, \mathbf{w}_\omega\right] \tag{IV.71}$$

where the intermediate path integrals now take the form

$$Z\left[\mathbf{A}, \mathbf{z}_\omega\right] = \int D\mathbf{X}_\omega \exp\left(\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\infty d\omega\left\{\mathbf{X}_\omega\cdot\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right)\mathbf{X}_\omega + \omega^2\mathbf{X}_\omega^2 - 2\left[\mathbf{z}_\omega - i\alpha\left(\mathbf{x},\omega\right)\frac{\partial\mathbf{A}}{\partial\tau}\right]\cdot\mathbf{X}_\omega\right\}\right) \tag{IV.72}$$

and for the magnetic field oscillator bath

$$Z\left[\mathbf{A}, \mathbf{w}_\omega\right] = \int D\mathbf{Y}_\omega \exp\left(\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\infty d\omega\left\{\mathbf{Y}_\omega\cdot\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right)\mathbf{Y}_\omega + \omega^2\mathbf{Y}_\omega^2 - 2\left[\mathbf{w}_\omega + \beta\left(\mathbf{x},\omega\right)\left(\nabla\times\mathbf{A}\right)\right]\cdot\mathbf{Y}_\omega\right\}\right). \tag{IV.73}$$

The two latter path integrals can be performed independently. Let's begin with the first of the two, $Z\left[\mathbf{A}, \mathbf{z}_\omega\right]$, and write it as

$$Z\left[\mathbf{A}, \mathbf{z}_\omega\right] = \int D\mathbf{X}_\omega \exp\left(\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\infty d\omega\left[\int_0^\beta d\tau'\,\mathbf{X}_\omega\left(\mathbf{x},\tau\right)\cdot\mathbf{D}_\omega\left(\tau,\tau'\right)\cdot\mathbf{X}_\omega\left(\mathbf{x},\tau'\right) - 2\,\mathbf{b}_\omega^{\mathbf{X}}\left(\mathbf{x},\tau\right)\cdot\mathbf{X}_\omega\left(\mathbf{x},\tau\right)\right]\right) \tag{IV.74}$$

where the matrix $\mathbf{D}_\omega(\tau,\tau')$ and the vector $\mathbf{b}^{\mathbf{X}}_\omega(\mathbf{x},\tau)$ are defined as

$$\begin{aligned}\mathbf{D}_\omega(\tau,\tau') &= \left(-\partial^2_{\tau'}+\omega^2\right)\delta(\tau-\tau') \\ \mathbf{b}^{\mathbf{X}}_\omega(\mathbf{x},\tau) &= \mathbf{z}_\omega(\mathbf{x},\tau) - i\alpha(\mathbf{x},\omega)\frac{\partial \mathbf{A}}{\partial\tau}(\mathbf{x},\tau)\,.\end{aligned} \tag{IV.75}$$

In order to perform the path integral, we introduce a shifted field

$$\mathbf{X}_\omega(\mathbf{x},\tau) \to \mathbf{X}_\omega(\mathbf{x},\tau) + \int_0^\beta d\tau''\,\mathbf{D}^{-1}_\omega(\tau,\tau'')\cdot\mathbf{b}^{\mathbf{X}}_\omega(\mathbf{x},\tau'')\,. \tag{IV.76}$$

Upon substitution of this shifted field, the path integral over the electric oscillator bath $\mathbf{X}_\omega$ can be written as being proportional to a term free of any dependence on this oscillator field

$$Z[\mathbf{A},\mathbf{z}_\omega] \propto \exp\left(\frac{i}{2\hbar}\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\beta d\tau'\int_0^\infty d\omega\,\mathbf{b}^{\mathbf{X}}_\omega(\mathbf{x},\tau)\cdot\mathbf{D}^{-1}_\omega(\tau,\tau')\cdot\mathbf{b}^{\mathbf{X}}_\omega(\mathbf{x},\tau')\right) \tag{IV.77}$$

providing that the condition on the $\mathbf{D}$ matrix

$$\int_0^\beta d\tau''\,\mathbf{D}_\omega(\tau,\tau'')\cdot\mathbf{D}^{-1}_\omega(\tau'',\tau') = \delta(\tau-\tau') \tag{IV.78}$$

is met. The proportionality constant is a path integral over an exponent quadratic in the oscillator field $\mathbf{X}_\omega$ but does not depend on the electromagnetic potential or the currents $\mathbf{z}_\omega$ and as such will eventually be removed by the normalisation of the full generating functional. The path integral over the oscillator bath for the magnetic field is analogous save for the $\mathbf{b}$ vector and reveals

$$Z[\mathbf{A},\mathbf{w}_\omega] \propto \exp\left(\frac{i}{2\hbar}\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\beta d\tau'\int_0^\infty d\omega\,\mathbf{b}^{\mathbf{Y}}_\omega(\mathbf{x},\tau)\cdot\mathbf{D}^{-1}_\omega(\tau,\tau')\cdot\mathbf{b}^{\mathbf{Y}}_\omega(\mathbf{x},\tau')\right) \tag{IV.79}$$

where now we have

$$\mathbf{b}^{\mathbf{Y}}_\omega(\mathbf{x},\tau) = \mathbf{w}_\omega(\mathbf{x},\tau) - \beta(\mathbf{x},\omega)\nabla\times\mathbf{A}(\mathbf{x},\tau)\,. \tag{IV.80}$$

In order to ascertain the form of the matrix $\mathbf{D}_{\omega}^{-1}(\tau'', \tau')$, the matrix $\mathbf{D}_{\omega}(\tau'', \tau')$ is substituted into (IV.78) to give

$$\left(-\partial_{\tau}^{2}+\omega^{2}\right)\mathbf{D}_{\omega}^{-1}(\tau,\tau')=\delta(\tau-\tau') \tag{IV.81}$$

where the delta function on the RHS is periodic in β. After expanding this periodic matrix into its Fourier components, one notices that this is just the Green's function for the harmonic oscillator in imaginary time

$$\mathbf{D}_{\omega}^{-1}(\tau,\tau')=G_{\omega}(\tau,\tau')\,\mathbb{1}_{3}=\frac{\mathbb{1}_{3}}{\beta}\sum_{n=-\infty}^{\infty}G_{\omega}(\omega_{n})\,e^{-i\omega_{n}(\tau-\tau')}. \tag{IV.82}$$

where $\omega_n = 2\pi n/\hbar\beta$ are the Matsubara frequencies and

$$G_{\omega}(\omega_{n})=\frac{1}{\omega^{2}+\omega_{n}^{2}}. \tag{IV.83}$$

Therefore we can write the two intermediate path integrals as

$$\begin{aligned}Z[\mathbf{A},\mathbf{z}_{\omega}]&\propto\exp\left(\frac{i}{2\hbar}\int d^{3}\mathbf{x}\int_{0}^{\beta}d\tau\int_{0}^{\beta}d\tau'\int_{0}^{\infty}d\omega\,G_{\omega}(\tau,\tau')\;\mathbf{b}_{\omega}^{\mathbf{X}}(\mathbf{x},\tau)\cdot\mathbf{b}_{\omega}^{\mathbf{X}}(\mathbf{x},\tau')\right)\\Z[\mathbf{A},\mathbf{w}_{\omega}]&\propto\exp\left(\frac{i}{2\hbar}\int d^{3}\mathbf{x}\int_{0}^{\beta}d\tau\int_{0}^{\beta}d\tau'\int_{0}^{\infty}d\omega\,G_{\omega}(\tau,\tau')\;\mathbf{b}_{\omega}^{\mathbf{Y}}(\mathbf{x},\tau)\cdot\mathbf{b}_{\omega}^{\mathbf{Y}}(\mathbf{x},\tau')\right).\end{aligned} \tag{IV.84}$$

Now, these expression must be substituted back into the the full path integral (IV.71), but first, let us make use of the expressions

$$\begin{aligned}&\int_{0}^{\beta}d\tau\int_{0}^{\beta}d\tau'\,G_{\omega}(\tau,\tau')\;\mathbf{b}_{\omega}^{\mathbf{X}}(\mathbf{x},\tau)\cdot\mathbf{b}_{\omega}^{\mathbf{X}}(\mathbf{x},\tau')=\int_{0}^{\beta}d\tau\int_{0}^{\beta}d\tau'\,G_{\omega}(\tau,\tau')\;\mathbf{z}_{\omega}(\mathbf{x},\tau)\cdot\mathbf{z}_{\omega}(\mathbf{x},\tau')\\&+i\int_{0}^{\beta}d\tau\int_{0}^{\beta}d\tau'\,\alpha(\mathbf{x},\omega)\frac{\partial}{\partial\tau}\left[G_{\omega}(\tau,\tau')+G_{\omega}(\tau',\tau)\right]\mathbf{A}(\mathbf{x},\tau)\cdot\mathbf{z}_{\omega}(\mathbf{x},\tau')\\&-\int_{0}^{\beta}d\tau\int_{0}^{\beta}d\tau'\,\alpha^{2}(\mathbf{x},\omega)\frac{\partial^{2}G_{\omega}(\tau,\tau')}{\partial\tau\partial\tau'}\mathbf{A}(\mathbf{x},\tau)\cdot\mathbf{A}(\mathbf{x},\tau')\end{aligned} \tag{IV.85}$$

and similarly for the magnetic case

$$\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\beta d\tau'\, G_\omega(\tau,\tau')\ \mathbf{b}_\omega^{\mathbf{Y}}(\mathbf{x},\tau)\cdot\mathbf{b}_\omega^{\mathbf{Y}}(\mathbf{x},\tau') =$$
$$\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\beta d\tau'\, G_\omega(\tau,\tau')\ \mathbf{w}_\omega(\mathbf{x},\tau)\cdot\mathbf{w}_\omega(\mathbf{x},\tau')$$
$$-\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\beta d\tau'\ \left[G_\omega(\tau,\tau')-G_\omega(\tau',\tau)\right]\nabla\times\left[\beta(\mathbf{x},\omega)\,\mathbf{w}_\omega(\mathbf{x},\tau')\right]\cdot\mathbf{A}(\mathbf{x},\tau)$$
$$-\int d^3\mathbf{x}\int_0^\beta d\tau\int_0^\beta d\tau'\, G_\omega(\tau,\tau')\ \beta^2(\mathbf{x},\omega)\,\mathbf{A}(\mathbf{x},\tau)\cdot\nabla\times\left[\nabla\times\mathbf{A}(\mathbf{x},\tau')\right].$$

(IV.86)

Using these expressions, the full path integral (IV.71) can be re-expressed as

$$Z[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega] = N\int D\mathbf{A}\ \exp\Bigg(-\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau\times$$
$$\int d^3\mathbf{x}'\int_0^\beta d\tau'\mathbf{A}(\mathbf{x},\tau)\cdot\mathbf{D}(\mathbf{x},\mathbf{x}',\tau,\tau')\cdot\mathbf{A}(\mathbf{x},\tau)+2\mathbf{b}(\mathbf{x},\tau)\cdot\mathbf{A}(\mathbf{x},\tau) \qquad \text{(IV.87)}$$
$$+\int_0^\beta d\tau'\int_0^\infty d\omega\, G_\omega(\tau,\tau')\ \left[\mathbf{z}_\omega(\mathbf{x},\tau)\cdot\mathbf{z}_\omega(\mathbf{x},\tau')+\mathbf{w}_\omega(\mathbf{x},\tau)\cdot\mathbf{w}_\omega(\mathbf{x},\tau')\right]\Bigg)$$

where this time the matrix $\mathbf{D}$ is given by

$$\mathbf{D}(\mathbf{x},\mathbf{x}',\tau,\tau') = \epsilon_0\left[\frac{\partial^2}{\partial\tau^2}+c^2\nabla'\times(\nabla'\times)\right]\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')$$
$$+\int_0^\infty d\omega\ \left[\alpha^2(\mathbf{x}',\omega)\,\frac{\partial^2 G_\omega(\tau,\tau')}{\partial\tau\partial\tau'}+\ G_\omega(\tau,\tau')\ \beta^2(\mathbf{x}',\omega)\,\nabla'\times(\nabla'\times)\right]\delta(\mathbf{x}-\mathbf{x}')$$

(IV.88)

and the $\mathbf{b}$ is defined as

$$\mathbf{b}(\mathbf{x},\tau) = \mathbf{j}(\mathbf{x},\tau) + \frac{1}{2}\int_0^\beta d\tau' \int_0^\infty d\omega \Bigg\{ i\alpha(\mathbf{x},\omega)\frac{\partial}{\partial\tau}\left[G_\omega(\tau,\tau') + G_\omega(\tau',\tau)\right]\cdot \mathbf{z}_\omega(\mathbf{x},\tau) \\ - \left[G_\omega(\tau,\tau') - G_\omega(\tau',\tau)\right]\nabla\times\left[\beta(\mathbf{x},\omega)\,\mathbf{w}_\omega(\mathbf{x},\tau')\right]\Bigg\}. \tag{IV.89}$$

Note that the last line of (IV.85) and (IV.86) end up combining with the free field wave equation to produce the electric and magnetic susceptibilities in (IV.88). We again proceed analogously to before and shift the electromagnetic potential field according to

$$\mathbf{A}(\mathbf{x},\tau) \to \mathbf{A}(\mathbf{x},\tau) + \int d^3\mathbf{x}'' \int_0^\beta d\tau''\, \mathbf{D}^{-1}(\mathbf{x},\mathbf{x}'',\tau,\tau'')\cdot \mathbf{b}(\mathbf{x},\tau'') \tag{IV.90}$$

which, providing the condition

$$\int d^3\mathbf{x}'' \int_0^\beta d\tau''\, \mathbf{D}(\mathbf{x},\mathbf{x}'',\tau,\tau'')\cdot \mathbf{D}^{-1}_\omega(\mathbf{x}'',\mathbf{x}',\tau'',\tau') = \delta(\mathbf{x}-\mathbf{x}')\,\delta(\tau-\tau') \tag{IV.91}$$

is met, and $\mathbf{b}$ is enforced to be periodic, leads to

$$Z[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega] = Z^{-1}[0,0,0]\,\exp\Bigg(-\frac{1}{2}\int d^3\mathbf{x}\int_0^\beta d\tau \times \\ \int d^3\mathbf{x}' \int_0^\beta d\tau'\, \mathbf{b}(\mathbf{x},\tau)\cdot \mathbf{D}^{-1}(\mathbf{x},\mathbf{x}',\tau,\tau')\cdot \mathbf{b}(\mathbf{x}',\tau') \\ + \int_0^\beta d\tau' \int_0^\infty d\omega\, G_\omega(\tau,\tau')\,\left[\mathbf{z}_\omega(\mathbf{x},\tau)\cdot\mathbf{z}_\omega(\mathbf{x},\tau') + \mathbf{w}_\omega(\mathbf{x},\tau)\cdot\mathbf{w}_\omega(\mathbf{x},\tau')\right]\Bigg). \tag{IV.92}$$

This time, the condition (IV.91) on the $\mathbf{D}^{-1}$ matrix reveals the imaginary time Green's function

$$\mathbf{D}^{-1}(\mathbf{x},\mathbf{x}',\tau,\tau') = \mathbf{G}(\mathbf{x},\mathbf{x}',\tau,\tau') = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} \mathbf{G}(\mathbf{x},\mathbf{x}',i\omega_n)\, e^{i\omega_n(\tau-\tau')} \tag{IV.93}$$

where ω_n are again the Matsubara frequencies and the Fourier components satisfy

$$\begin{aligned}\nabla\times\left[\left(1+\int_0^\infty d\omega\frac{\beta^2\left(\mathbf{x},\omega\right)}{\omega^2+\omega_n^2}\right)\nabla\times\mathbf{G}\left(\mathbf{x},\mathbf{x}',i\omega_n\right)\right]\\ +\frac{\omega_n^2}{c^2}\left[1+\int_0^\infty d\omega\frac{\alpha^2\left(\mathbf{x},\omega\right)}{\omega^2+\omega_n^2}\right]\mathbf{G}\left(\mathbf{x},\mathbf{x}',i\omega_n\right)=\delta\left(\mathbf{x}-\mathbf{x}'\right).\end{aligned} \tag{IV.94}$$

Note the appearance of the imaginary frequency permittivity

$$\epsilon(i\omega_n)=1+\int_0^\infty d\omega\,\frac{\alpha^2\left(\mathbf{x},\omega\right)}{\omega^2+\omega_n^2} \tag{IV.95}$$

analogous to (I.17) and imaginary frequency inverse permeability $\mu^{-1}(i\omega_n) = 1+\int_0^\infty d\omega\,\beta^2\left(\mathbf{x},\omega\right)/(\omega^2+\omega_n^2)$. In fact, this is just the wave equation for the electromagnetic Green's function (III.30) evaluated at imaginary frequencies.

It is worth remarking that the effect of integrating over the oscillator fields is essentially to remove the microscopic degrees of freedom of the material and leaves us with a theory of electromagnetism within a magneto-dielectric with no explicit reference to the microscopic structure of the matter responsible for the permittivity and permeability.

In order to extract the thermal correlation functions, we proceed as before by taking functional derivatives with respect to the various currents $\mathbf{j}$, $\mathbf{z}_\omega$ or $\mathbf{w}_\omega$. The derivative of a functional, essentially a function of a function, are defined in an analogous manner to the usual derivatives. For example, consider a functional $I[f(x)]$ of some function $f(x)$, the derivative of the functional I with respect to particular function $f(x)$ is defined by

$$\frac{\partial I\left[f(x)\right]}{\partial f(y)}=\lim_{\eta\to 0}\frac{I\left[f(x)+\eta\delta(x-y)\right]-I\left[f(x)\right]}{\eta}. \tag{IV.96}$$

For example, suppose $I[j(x)] = \int_{-\infty}^{\infty} dx'\,G(x,x')j(x')$, then the derivative with respect to the function j at a given point would be

$$\begin{aligned}\frac{\partial I\left[j(x)\right]}{\partial j(y)} &= \lim_{\eta\to 0}\frac{\int_{-\infty}^{\infty}dx'\,G(x,x')\left[j(x')+\eta\delta(x'-y)\right]-\int_{-\infty}^{\infty}dx\,G(x,x')j(x')}{\eta}\\ &= \lim_{\eta\to 0}\frac{\int_{-\infty}^{\infty}dx'\,G(x,x')\eta\delta(x'-y)+\int_{-\infty}^{\infty}dx'\,G(x,x')j(x')-\int_{-\infty}^{\infty}dx'\,G(x,x')j(x')}{\eta}\\ &= \int_{-\infty}^{\infty}dx'\,G(x,x')\delta(x'-y)=G(x,y)\end{aligned} \tag{IV.97}$$

Using this definition, we may extract correlation functions from the generating functional. As an example, for the two point function for the electric field $\langle\mathbf{E}(\tau)\otimes\mathbf{E}(\tau')\rangle$, by considering that $\mathbf{E}=-\partial_t\mathbf{A}$, the correlation function ends up being $\langle\mathbf{E}(\tau)\otimes\mathbf{E}(\tau')\rangle=\partial_\tau\partial_{\tau'}\langle\mathbf{A}(\tau)\otimes\mathbf{A}(\tau')\rangle$. In terms of derivatives of currents, this is

$$\langle\mathbf{A}(\mathbf{x},\tau)\otimes\mathbf{A}(\mathbf{x}',\tau')\rangle=\left(\frac{\hbar}{i}\right)^2\frac{\partial^2 Z\left[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega\right]}{\partial\mathbf{j}(\mathbf{x},\tau)\otimes\partial\mathbf{j}(\mathbf{x}',\tau')}\bigg|_{\mathbf{j}=\mathbf{z}_\omega=\mathbf{w}_\omega=0}. \tag{IV.98}$$

Using the Lorentz reciprocity $\mathbf{G}\left(\mathbf{x},\mathbf{x}',i\omega_n\right)=\mathbf{G}\left(\mathbf{x}',\mathbf{x},i\omega_n\right)$, the correlation function for the electric field is thus expressed as

$$\langle\mathbf{E}(\mathbf{x},\tau)\otimes\mathbf{E}(\mathbf{x}',\tau)\rangle=-\frac{2\omega_n^2}{\beta}\sum_{n=1}^{\infty}\mathbf{G}\left(\mathbf{x},\mathbf{x}',i\omega_n\right) \tag{IV.99}$$

which, again using the poles of the hyperbolic cotangent can be written as a contour integral. This time we take a slightly different approach though, and use the red contour of Figure IV.3) which can be deformed into the blue contour. Noting that the integrand goes to zero as $|\omega|\to\infty$ yields the usual fluctuation-dissipation result,

$$\langle\mathbf{E}(\mathbf{x},\tau)\otimes\mathbf{E}(\mathbf{x}',\tau)\rangle=\frac{\hbar\mu_0}{\pi}\int_0^{\infty}d\omega\,\omega^2\coth\left(\frac{\hbar\omega}{2k_BT}\right)\mathrm{Im}\left[\mathbf{G}(\mathbf{x},\mathbf{x}',\omega)\right]. \tag{IV.100}$$

One can of course take derivatives with respect to the currents $\mathbf{z}_\omega$ or $\mathbf{w}_\omega$ to get combinations involving the oscillator bath fields $\mathbf{X}_\omega$ and $\mathbf{Y}_\omega$ respectively.

Here, we have shown that is possible to derive the thermal expectation values

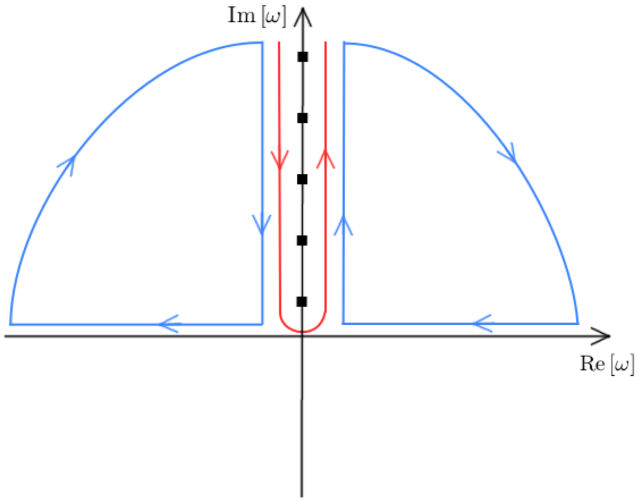

Figure IV.3: The black squares represent the Matsubara frequencies $i\omega_n$. The sum over the Green's function evaluated at the Matsubara frequencies can be written as the red contour where the hyperbolic cotangent is introduced since its residues pick out the Green's function evaluated at those frequencies. This contour can be deformed to become the blue contour the integral resulting in the usual fluctuation-dissipation result.

of the field amplitudes in macroscopic QED for time-independent systems using the Matsubara (imaginary time) formulation of the path integral. Our ultimate goal however, is to calculate the changes in the thermal expectation values in systems subject to an explicit time dependence. In the above expressions however, the reader will notice that there is no reference to a real time variable, only imaginary. Let us now consider the real time formulation of the path integral applied to macroscopic QED.

IV.3 Real Time Generating Functional

The concept of the generating functional is an elegant way of arriving at thermal expected values without needing to resort to the operator formalism of the quantum theory. In the previous section, we demonstrated for the first time that it can be applied to macroscopic QED in order to recover the thermal two point functions such as (III.159). However, our ultimate goal is to understand how the vacuum field is modified by the presence of time-dependent dielectrics, either via their non-uniform motion, an explicit time-dependence of the electromagnetic response functions or a combination of both. As the reader will have noticed, the formulation for deriving equilibrium thermal correlation functions using the path integral contains only purely imaginary times. In order to take into account the explicit time dependencies of the medium, let us now explore the equivalent real-time formalism extensively used in quantum field theory. Again, as far as the author is aware, this has not yet been derived for this formulation of macroscopic QED.

IV.3.1 Generating Functional for Time-independent Media

In the spirit of treating the simplest cases first, before treating the more involved path integral formalism for time-dependent Lagrangians, let us begin by demonstrating the real time formulation of the path integral method applied to the time independent macroscopic QED system of Section III.2. The approach taking here bears some resemblance to that of [100] in that we will 'integrate out' the degrees of freedom of the matter field (as in the thermal case). However, the Lagrangian used was that of [88] rather than that of [84].

Defining a new, real time generating functional

$$\begin{aligned} Z\left[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega\right] = \int D\mathbf{A}\int D\mathbf{X}_\omega\int D\mathbf{Y}_\omega \exp\left\{\frac{i}{\hbar}S\left[\mathbf{A},\mathbf{X}_\omega,\mathbf{Y}_\omega\right]\right\} \\ \times\exp\left\{\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\left[\mathbf{A}\cdot\mathbf{j}+\int_0^\infty d\omega\,\left(\mathbf{X}_\omega\cdot\mathbf{z}_\omega+\mathbf{Y}_\omega\cdot\mathbf{w}_\omega\right)\right]\right\} \end{aligned} \tag{IV.101}$$

where the action is now given in terms of the real time Lagrangian (IV.66) $\mathscr{L}_0$ as

$$S\left[\mathbf{A},\mathbf{X}_\omega,\mathbf{Y}_\omega\right] = \int d^4x\ \mathscr{L}_0\left(\mathbf{A},\mathbf{X}_\omega,\mathbf{Y}_\omega\right) \tag{IV.102}$$

allows us to carry out the path integral in a manner analogous to the previous section with only minor modifications. Firstly, the most obvious, is that the integral over time is no longer imaginary and periodic but is real and extends from $-\infty$ to ∞. The result of this is that the generating functional written in terms of the intermediary path integrals is written as

$$Z[\mathbf{j}, \mathbf{z}_\omega, \mathbf{w}_\omega] = \int D\mathbf{A} \, \exp\left(\frac{i}{2\hbar}\int d^3\mathbf{x} \int\limits_{-\infty}^{\infty} dt \left\{\epsilon_0 \left[\dot{\mathbf{A}}^2 - c^2 (\nabla \times \mathbf{A})^2\right] + 2\mathbf{A}\cdot\mathbf{j}\right\}\right) \times Z[\mathbf{A}, \mathbf{z}_\omega] \, Z[\mathbf{A}, \mathbf{w}_\omega] \tag{IV.103}$$

where the intermediary integral over the oscillator bath for the electric field is

$$Z[\mathbf{A}, \mathbf{z}_\omega] = \int D\mathbf{X}_\omega \times \exp\left(\frac{i}{2\hbar}\int d^3\mathbf{x} \int\limits_{-\infty}^{\infty} dt \int\limits_{0}^{\infty} d\omega \left\{\dot{\mathbf{X}}_\omega^2 - \omega^2 \mathbf{X}_\omega^2 + 2\left[\mathbf{z}_\omega - \alpha(\mathbf{x}, \omega)\,\dot{\mathbf{A}}\right]\cdot\mathbf{X}_\omega\right\}\right) \tag{IV.104}$$

and the path integral over the oscillator bath for the magnetic field is

$$Z[\mathbf{A}, \mathbf{w}_\omega] = \int D\mathbf{Y}_\omega \times \exp\left(\frac{i}{2\hbar}\int d^3\mathbf{x} \int\limits_{-\infty}^{\infty} dt \int\limits_{0}^{\infty} d\omega \left\{\dot{\mathbf{Y}}_\omega^2 - \omega^2 \mathbf{Y}_\omega^2 + 2\left[\mathbf{w}_\omega + \beta(\mathbf{x}, \omega)(\nabla \times \mathbf{A})\right]\cdot\mathbf{Y}_\omega\right\}\right). \tag{IV.105}$$

We can perform the path integral in a similar manner by shifting the oscillator fields $\mathbf{X}_\omega$ and $\mathbf{Y}_\omega$ as before via

$$\begin{aligned} \mathbf{X}_\omega(\mathbf{x}, t) &\to \mathbf{X}_\omega(\mathbf{x}, t) + \int\limits_{-\infty}^{\infty} dt'' \, \mathbf{D}_\omega^{-1}(t, t'')\cdot\mathbf{b}_\omega^{\mathbf{X}}(\mathbf{x}, t'') \\ \mathbf{Y}_\omega(\mathbf{x}, t) &\to \mathbf{Y}_\omega(\mathbf{x}, t) + \int\limits_{-\infty}^{\infty} dt'' \, \mathbf{D}_\omega^{-1}(t, t'')\cdot\mathbf{b}_\omega^{\mathbf{Y}}(\mathbf{x}, t'') \end{aligned} \tag{IV.106}$$

where the vectors $\mathbf{b}_\omega^{\mathbf{X}}$ and $\mathbf{b}_\omega^{\mathbf{Y}}$ are now given by

$$\begin{aligned} \mathbf{b}_\omega^{\mathbf{X}}(\mathbf{x},t) &= \mathbf{z}_\omega(\mathbf{x},t) - \alpha(\mathbf{x},\omega)\,\dot{\mathbf{A}}(\mathbf{x},t) \\ \mathbf{b}_\omega^{\mathbf{Y}}(\mathbf{x},t) &= \mathbf{w}_\omega(\mathbf{x},t) + \beta(\mathbf{x},\omega)\,\nabla\times\mathbf{A}(\mathbf{x},t)\,. \end{aligned} \tag{IV.107}$$

Thus, the intermediary path integrals are expressed as

$$\begin{aligned} Z[\mathbf{A},\mathbf{z}_\omega] &\propto \exp\left(\frac{i}{2\hbar}\int d^3\mathbf{x}\int dt\int dt'\int_0^\infty d\omega\, M_z(\mathbf{x},t,t')\right) \\ Z[\mathbf{A},\mathbf{z}_\omega] &\propto \exp\left(\frac{i}{2\hbar}\int d^3\mathbf{x}\int dt\int dt'\int_0^\infty d\omega\, M_w(\mathbf{x},t,t')\right) \end{aligned} \tag{IV.108}$$

where the exponents are defined as

$$\begin{aligned} M_z(\mathbf{x},t,t') &= \alpha(\mathbf{x},\omega)\,\partial_t\left[G_\omega(t-t')+G_\omega(t'-t)\right]\mathbf{A}(\mathbf{x},t)\cdot\mathbf{z}_\omega(\mathbf{x},t') \\ &\quad + G_\omega(t-t')\,\mathbf{z}_\omega(\mathbf{x},t)\cdot\mathbf{z}_\omega(\mathbf{x},t') \\ &\quad + \alpha^2(\mathbf{x},\omega)\,\partial_t\partial_{t'}G_\omega(t-t')\,\mathbf{A}(\mathbf{x},t)\cdot\mathbf{A}(\mathbf{x},t') \end{aligned} \tag{IV.109}$$

for the electric oscillator bath and

$$\begin{aligned} M_w(\mathbf{x},t,t') &= G_\omega(t-t')\ \mathbf{w}_\omega(\mathbf{x},t)\cdot\mathbf{w}_\omega(\mathbf{x},t') \\ &\quad + \left[G_\omega(t-t')+G_\omega(t'-t)\right]\nabla\times\left[\beta(\mathbf{x},\omega)\,\mathbf{w}_\omega(\mathbf{x},t)\right]\cdot\mathbf{A}(\mathbf{x},t') \\ &\quad + G_\omega(t-t')\ \nabla\times\left[\beta^2(\mathbf{x},\omega)\,\nabla\times\mathbf{A}(\mathbf{x},t)\right]\cdot\mathbf{A}(\mathbf{x},t') \end{aligned} \tag{IV.110}$$

for the magnetic field bath. The Green's function $G_\omega(t-t')$ is nothing more than the usual harmonic oscillator Green's function satisfying

$$\left(\partial_t^2+\omega^2\right)G_\omega(t,t') = \delta(t-t')\,. \tag{IV.111}$$

Here, we pick the boundary conditions such that it is the retarded solutions that are retrieved. Again, the full generating functional is performed as before and can be found to be

$$Z\left[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega\right]=Z^{-1}\left[0,0,0\right]\exp\left(\frac{i}{2\hbar}\int d^3\mathbf{x}\int dt\int dt'\times\right.$$
$$\int d^3\mathbf{x}'\,\mathbf{b}\left(\mathbf{x},t\right)\cdot\mathbf{D}^{-1}\left(\mathbf{x},\mathbf{x}',t,t'\right)\cdot\mathbf{b}\left(\mathbf{x}',t'\right)$$
$$\left.+\int_0^\infty d\omega\,G_\omega\left(t-t'\right)\left[\mathbf{w}_\omega\left(\mathbf{x},t\right)\cdot\mathbf{w}_\omega\left(\mathbf{x},t'\right)+\mathbf{z}_\omega\left(\mathbf{x},t\right)\cdot\mathbf{z}_\omega\left(\mathbf{x},t'\right)\right]\right) \tag{IV.112}$$

where the matrix $\mathbf{D}$ is given by

$$\begin{aligned}\mathbf{D}\left(\mathbf{x},\mathbf{x}',t,t'\right)=&\left[\epsilon_0\partial_{t'}^2-\epsilon_0c^2\nabla'\times\left(\nabla'\times\right)\right]\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right)\\&-\int_0^\infty d\omega\alpha^2\left(\mathbf{x}',\omega\right)\partial_t\partial_{t'}G_\omega\left(t-t'\right)\delta\left(\mathbf{x}-\mathbf{x}'\right)\\&+\int_0^\infty d\omega G_\omega\left(t-t'\right)\,\nabla'\times\left(\beta^2\left(\mathbf{x}',\omega\right)\nabla'\times\right)\delta\left(\mathbf{x}-\mathbf{x}'\right)\end{aligned} \tag{IV.113}$$

and the vector $\mathbf{b}$ is given by

$$\begin{aligned}\mathbf{b}\left(\mathbf{x},t\right)=\mathbf{j}\left(\mathbf{x},t\right)+\frac{1}{2}\int dt'\int_0^\infty d\omega&\left\{\alpha\left(\mathbf{x},\omega\right)\partial_t\left[G_\omega\left(t-t'\right)+G_\omega\left(t'-t\right)\right]\mathbf{z}_\omega\left(\mathbf{x},t'\right)\right.\\&\left.+\left[G_\omega\left(t-t'\right)+G_\omega\left(t'-t\right)\right]\nabla\times\left[\beta\left(\mathbf{x},\omega\right)\mathbf{w}_\omega\left(\mathbf{x},t'\right)\right]\right\}.\end{aligned} \tag{IV.114}$$

This time, the condition on the inverse matrix $\mathbf{D}^{-1}$ is

$$\int d^3\mathbf{x}''\int_{-\infty}^{\infty}dt''\,\mathbf{D}\left(\mathbf{x},\mathbf{x}'',t,t''\right)\cdot\mathbf{D}^{-1}\left(\mathbf{x}'',\mathbf{x}',t'',t'\right)=\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right). \tag{IV.115}$$

This leads to the wave equation for $\mathbf{G}\left(\mathbf{x},\mathbf{x}',t,t'\right)$, which in the frequency domain, is of course (III.30). We can now take derivatives with respect to the currents to get various real time correlations functions in analogy to our previous thermal results. Just as in usual vacuum QFT, these are the time-ordered correlations functions. For example for the vector potential, we would obtain, using the Lorentz reciprocity condition $\mathbf{G}(\mathbf{x},\mathbf{x}',\omega)=\mathbf{G}(\mathbf{x}',\mathbf{x},\omega)$

$$\begin{aligned}\langle\hat{T}[\hat{\mathbf{A}}(t)\otimes\hat{\mathbf{A}}(t')]\rangle &= \left(\frac{\hbar}{i}\right)^2 \frac{\partial^2 Z\left[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega\right]}{\partial\mathbf{j}(\mathbf{x},t)\otimes\partial\mathbf{j}(\mathbf{x}',t')}\bigg|_{\mathbf{j}=\mathbf{z}_\omega=\mathbf{w}_\omega=0} \\ &= \frac{1}{2}\left[\mathbf{G}(\mathbf{x},\mathbf{x}',t,t')+\mathbf{G}(\mathbf{x}',\mathbf{x},t',t)\right] \\ &= \int_{-\infty}^{\infty} d\omega\,\mathrm{Re}\left[\mathbf{G}(\mathbf{x},\mathbf{x}',\omega)\right]e^{-i\omega(t-t')}\end{aligned} \tag{IV.116}$$

which can be seen to be half of the sum of the advanced and retarded solutions to the wave equation (III.30) in a similar fashion to the usual QFT path integral [49]. Now we wish to consider the same real time path integral but for systems containing an arbitrary time dependence such as those explored in Section III.4.

IV.3.2 Generating Functional for Time-dependent Media

As previously discussed, we are often interested in dynamical situations where some external force is creating a local time dependence of the Lagrangian such as those explored in section III.4.

The correct starting point for performing path integrals is the general expression (IV.31). For the vast majority of systems, this turns out to be equivalent to the path integral expressed in terms of the Lagrangian as in (IV.32). However, because of the explicit time dependence of the Hamiltonian of Section III.4, the correspondence is no longer a given and one might ask whether the two are now really equivalent. Perhaps the correct path integral should be the more general (IV.31) expressed in terms of the canonical momenta.

Validity of the Lagrangian Formulation for Time-dependent Hamiltonians

To understand whether we can use (IV.32) in the time dependent case, we must consider how one arrives at (IV.32) from (IV.31) in the simple time independent case and see how this derivation changes in the time varying case. Consider the slightly simpler Lagrangian density for a scalar field ϕ coupled to one bath of harmonic oscillators X_ω

$$\mathscr{L} = \frac{1}{2}\left[-\left(\nabla\phi\right)^2 + \frac{1}{c^2}\left(\dot{\phi}\right)^2\right] - \partial_t\phi \cdot \int\limits_0^\infty d\omega\ \alpha\left(\mathbf{x},\omega\right)\mathsf{X}_\omega + \frac{1}{2}\int\limits_0^\infty d\omega\left[\left(\partial_t\mathsf{X}_\omega\right)^2 - \omega^2\mathsf{X}_\omega^2\right]. \tag{IV.117}$$

In this case, the canonical momenta can be found

$$\begin{aligned}\Pi_\phi &= \frac{\partial\mathscr{L}}{\partial\left(\partial_t\phi\right)} = \frac{1}{c^2}\partial_t\phi - \int\limits_0^\infty d\omega\ \alpha\left(\mathbf{x},\omega\right)\mathsf{X}_\omega \\ \Pi_{\mathsf{X}_\omega} &= \frac{\partial\mathscr{L}}{\partial\left(\partial_t\mathsf{X}_\omega\right)} = \partial_t\mathsf{X}_\omega.\end{aligned} \tag{IV.118}$$

One can then derive the Hamiltonian in the same way as section III.2 by computing

$$\begin{aligned}\mathscr{H} &= \Pi_\phi\cdot\dot{\phi} + \int\limits_0^\infty d\omega\ \Pi_{\mathsf{X}_\omega}\cdot\dot{\mathsf{X}}_\omega - \mathscr{L} \\ \mathscr{H} &= \frac{1}{2}c^2\Pi_\phi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + c^2\Pi_\phi\cdot\int\limits_0^\infty d\omega\ \alpha\left(\mathbf{x},\omega\right)\mathsf{X}_\omega \\ &\quad + \frac{1}{2}c^2\int\limits_0^\infty d\omega\ \alpha\left(\mathbf{x},\omega\right)\mathsf{X}_\omega\cdot\int\limits_0^\infty d\omega\ \alpha\left(\mathbf{x},\omega\right)\mathsf{X}_\omega + \frac{1}{2}\int\limits_0^\infty d\omega\ \left(\Pi_{\mathsf{X}_\omega}^2 + \omega^2\mathsf{X}_\omega^2\right).\end{aligned} \tag{IV.119}$$

This allows us to define a path integral based on the definition (IV.31) in terms of the Hamiltonian and the canonical momenta

$$K = \int D\phi\int D\Pi_\phi\int D\mathsf{X}_\omega\int D\Pi_{\mathsf{X}_\omega}\exp\left(\frac{i}{\hbar}S\left[\phi,\Pi_\phi,\mathsf{X}_\omega,\Pi_{\mathsf{X}_\omega}\right]\right) \tag{IV.120}$$

where the action as a functional of the canonical variables and their momenta is given by

$$S\left[\phi,\Pi_\phi,\mathsf{X}_\omega,\Pi_{\mathsf{X}_\omega}\right] = \int d^3\mathbf{x}\int dt\left\{\Pi_\phi\cdot\dot{\phi} + \int\limits_0^\infty d\omega\ \Pi_{\mathsf{X}_\omega}\cdot\dot{\mathsf{X}}_\omega - \mathscr{H}\left[\phi,\Pi_\phi,\mathsf{X}_\omega,\Pi_{\mathsf{X}_\omega}\right]\right\}. \tag{IV.121}$$

To arrive at the Lagrangian formalism (IV.32), we must proceed analogously to the simple quantum mechanical case and perform the path integrals over the momenta of the scalar field and oscillator field. We can write the problem in a somewhat simpler form by again writing it as two path integrals to be performed one after the other

$$K=\int D\phi\int D\Pi_{\phi}\exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\left\{\Pi_{\phi}\cdot\dot{\phi}-\frac{1}{2}c^2\Pi_{\phi}^2-\frac{1}{2}\left(\nabla\phi\right)^2\right\}\right)K_{\omega}\left[\phi,\Pi_{\phi}\right] \tag{IV.122}$$

where we define the intermediate path integral in terms of the canonical variables and their momenta as

$$\begin{aligned}
&K_{\omega}\left[\phi,\Pi_{\phi}\right]=\int D\mathsf{X}_{\omega}\times\\
&\exp\left(-\frac{i}{2\hbar}\int d^3\mathbf{x}\int dt\int_0^{\infty}d\omega\left\{c^2\left[2\Pi_{\phi}\cdot+\int_0^{\infty}d\omega'\;\alpha\left(\mathbf{x},\omega'\right)\mathsf{X}_{\omega'}\right]\cdot\alpha\left(\mathbf{x},\omega\right)\mathsf{X}_{\omega}+\omega^2\mathsf{X}_{\omega}^2\right\}\right)\\
&\times\int D\Pi_{\mathsf{X}_{\omega}}\exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\int_0^{\infty}d\omega\left\{-\frac{1}{2}\Pi_{\mathsf{X}_{\omega}}^2+\Pi_{\mathsf{X}_{\omega}}\cdot\dot{\mathsf{X}}_{\omega}\right\}\right).
\end{aligned} \tag{IV.123}$$

Performing first of all the integral over the canonical momentum of the oscillator field

$$\begin{aligned}
&K_{\omega}\left[\phi,\Pi_{\phi}\right]\propto\int D\mathsf{X}_{\omega}\exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\int_0^{\infty}d\omega\times\right.\\
&\left.\left\{-c^2\Pi_{\phi}\cdot\alpha\left(\mathbf{x},\omega\right)\mathsf{X}_{\omega}-\frac{1}{2}c^2\alpha\left(\mathbf{x},\omega\right)\mathsf{X}_{\omega}\cdot\int_0^{\infty}d\omega'\;\alpha\left(\mathbf{x},\omega'\right)\mathsf{X}_{\omega'}+\frac{1}{2}\left(\dot{\mathsf{X}}_{\omega}\right)^2-\frac{1}{2}\omega^2\mathsf{X}_{\omega}^2\right\}\right).
\end{aligned} \tag{IV.124}$$

where we have used the identity

$$\prod_i^N\left(\int dx_i\right)\;e^{-\frac{1}{2}\mathbf{x}^T\cdot\mathbf{A}\cdot\mathbf{x}+\mathbf{b}\cdot\mathbf{x}}=\frac{1}{\sqrt{\det\mathbf{A}}}e^{\frac{1}{2}\mathbf{b}^T\cdot\mathbf{A}^{-1}\cdot\mathbf{b}} \tag{IV.125}$$

which is an analogous way of performing the path integral to using the change of variables such as that in (IV.106). The full path integral then becomes

$$
\begin{aligned}
&K \propto \int D\phi \int D\mathsf{X}_\omega \times \\
&\exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\left\{-\frac{1}{2}(\nabla\phi)^2+\int_0^\infty d\omega\left[-\frac{1}{2}c^2\alpha(\mathbf{x},\omega)\mathsf{X}_\omega\cdot\int_0^\infty d\omega'\,\alpha(\mathbf{x},\omega')\mathsf{X}_{\omega'}\right.\right.\right. \\
&\qquad\qquad\qquad\left.\left.\left.+\frac{1}{2}\left(\dot{\mathsf{X}}_\omega\right)^2-\frac{1}{2}\omega^2\mathsf{X}_\omega^2\right]\right\}\right) \\
&\qquad\times\int D\Pi_\phi\exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\left\{-\frac{1}{2}c^2\Pi_\phi^2+\Pi_\phi\cdot\left[\dot{\phi}-c^2\int_0^\infty d\omega\alpha(\mathbf{x},\omega)\mathsf{X}_\omega\right]\right\}\right)
\end{aligned}
\tag{IV.126}
$$

Performing the integral over the scalar momentum Π_ϕ then yields

$$
\begin{aligned}
K &\propto \int D\phi\int D\mathsf{X}_\omega\exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\left\{\frac{1}{2}\left[-(\nabla\phi)^2+\frac{1}{c^2}\left(\dot{\phi}\right)^2\right]\right.\right. \\
&\qquad\left.\left.-\dot{\phi}\int_0^\infty d\omega\alpha(\mathbf{x},\omega)\mathsf{X}_\omega+\frac{1}{2}\int_0^\infty d\omega\left[(\partial_t\mathsf{X}_\omega)^2-\omega^2\mathsf{X}_\omega^2\right]\right\}\right) \qquad \text{(IV.127)} \\
&\propto \int D\phi\int D\mathsf{X}_\omega\exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\,\mathscr{L}\right)
\end{aligned}
$$

which is just the path integral (IV.32). The question is now, how is this derivation modified by the presence of the time dependence contained in the Lagrangian density (III.113). Again, let's consider the simpler problem where all the time dependence comes from the time dependent motion of the oscillator field X_ω, again coupled to a scalar field ϕ. The Lagrangian density in this case is

$$
\begin{aligned}
\mathscr{L} &= \frac{1}{2}\left[-(\nabla\phi)^2+\frac{1}{c^2}\left(\dot{\phi}\right)^2\right]-\left[\partial_t-\mathbf{v}(t)\cdot\nabla\right]\phi\cdot\int_0^\infty d\omega\,\alpha(\mathbf{x},\omega)\mathsf{X}_\omega \\
&\qquad+\frac{1}{2}\int_0^\infty d\omega\left\{\left[\left(\partial_t-\mathbf{v}(t)\cdot\nabla\right)\mathsf{X}_\omega\right]^2-\omega^2\mathsf{X}_\omega^2\right\}
\end{aligned}
\tag{IV.128}
$$

and then canonical momenta are no longer the same, they are now found to be

$$\begin{aligned}\Pi_\phi &= \frac{\partial \mathscr{L}}{\partial\left(\partial_t \phi\right)} = \frac{1}{c^2}\partial_t \phi - \int_0^\infty d\omega\ \alpha\left(\mathbf{x}, \omega\right) \mathsf{X}_\omega \\ \Pi_{\mathsf{X}_\omega} &= \frac{\partial \mathscr{L}}{\partial\left(\partial_t \mathsf{X}_\omega\right)} = \left[\partial_t - \mathbf{v}\left(t\right)\cdot\nabla\right] \mathsf{X}_\omega.\end{aligned} \tag{IV.129}$$

In terms of these new quantities, the Hamiltonian density is found to be the scalar form of that of Section III.4 as expected

$$\mathscr{H} = \mathscr{H}_0 - \int_0^\infty d\omega\ \left\{\alpha\left(\mathbf{x}, \omega\right)\left[\mathbf{v}\left(t\right)\cdot\nabla\right]\phi \cdot \mathsf{X}_\omega + \Pi_{\mathsf{X}_\omega}\cdot\left(\mathbf{v}\cdot\nabla\right)\mathsf{X}_\omega\right\} \tag{IV.130}$$

where the part that is independent of the velocity $\mathscr{H}_0$ is that of (IV.119). Again, our starting point is the path integral (IV.120), and we must perform the path integrals over the canonical momenta Π_ϕ and Π_{X_ω} as before. Attending to the momentum of the oscillator field first, we find that we encounter a path integral of the form

$$\int D\Pi_{\mathsf{X}_\omega} \exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\int_0^\infty d\omega\left\{-\frac{1}{2}\,\Pi_{\mathsf{X}_\omega}^2 + \Pi_{\mathsf{X}_\omega}\cdot\left\{\dot{\mathsf{X}}_\omega + \left[\mathbf{v}\left(t\right)\cdot\nabla\right]\mathsf{X}_\omega\right\}\right\}\right) \tag{IV.131}$$

instead of the stationary case equivalent

$$\int D\Pi_{\mathsf{X}_\omega} \exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\int_0^\infty d\omega\left[-\frac{1}{2}\,\Pi_{\mathsf{X}_\omega}^2 + \Pi_{\mathsf{X}_\omega}\cdot\dot{\mathsf{X}}_\omega\right]\right) \tag{IV.132}$$

and there seems to be no reason why this path integral cannot be performed in exactly the same manner, at least for slow well behaved velocities. The integral over the momentum of the scalar field turns out to be exactly the same as the stationary case here and therefore needs no extra consideration. We therefore arrive at

$$K \propto \int D\phi \int \mathsf{X}_\omega \exp\left(\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\ \mathscr{L}\right) \tag{IV.133}$$

The same arguments hold for the other forms of time-dependence that we will consider and we conclude that the Lagrangian formulation of the path integral is a suitable starting point for our calculations. In theory at least, we can proceed analogously to the previous section.

A Simple Example: Parallel Plates

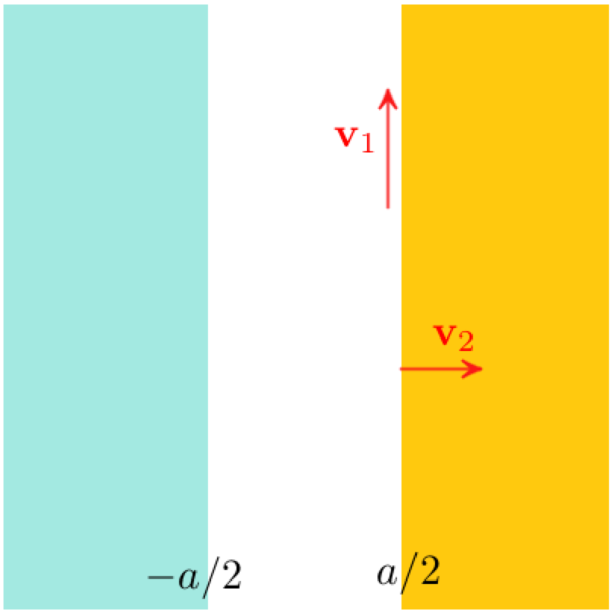

Figure IV.4: Two infinite half space dielectric plates separated by the vacuum moving with either $\mathbf{v}_1$ parallel to the surface of the plates or $\mathbf{v}_2$ perpendicular to the surface of the plates.

Now that we've shown that the Lagrangian formulation (IV.32) of the path integral is a correct starting point, let's try to use the same techniques used for the stationary case in the previous section by considering the simple case of two parallel half plane dielectrics separated by a distance a and moving with a relative motion $\mathbf{v} = v\hat{\mathbf{y}}$ in y-direction such as in Figure IV.4. In order to consider the path integral, the simplest approach is to consider both dielectrics as containing their own oscillators fields, $\mathsf{X}_{L\omega}$ for the dielectric occupying the space $[-\infty, -a/2]$ and $\mathsf{X}_{R\omega}$ occupying the space $[a/2, \infty]$. Then we can define the path integral as before but using *two* intermediate generating functionals (one for the moving medium and one for the medium at rest)

$$Z\left[j\right] = \int D\phi \, \exp\left(-\frac{i}{2\hbar}\int d^3\mathbf{x}\int_{-\infty}^{\infty} dt\left[-\left(\nabla\phi\right)^2 + \frac{1}{c^2}\left(\dot{\phi}\right)^2 - 2\phi\cdot j\right]\right) Z^L_{\mathsf{X}}\left[\phi\right] Z^R_{\mathsf{X}}\left[\phi\right] \tag{IV.134}$$

where the intermediary path integrals are defined as

$$Z_X^\lambda(\phi) = \int DX_{\lambda\omega} \exp\Bigg(-\frac{i}{2\hbar}\int d^3\mathbf{x} \int_{-\infty}^{\infty} dt \int_0^{\infty} d\omega \times \\ \left[\left(\partial_t^\lambda X_{\lambda\omega}\right)\cdot\left(\partial_t^\lambda X_{\lambda\omega}\right) + \omega^2 X_{\lambda\omega}\cdot X_{\lambda\omega} + 2\alpha_\lambda(x,\omega)\,\partial_t\phi\cdot X_{\lambda\omega}\right]\Bigg) \tag{IV.135}$$

where $\lambda = L, R$ and in the moving medium, the derivatives with respect to time of the oscillator amplitudes take the form $\partial_t^L = \partial_t - v\partial_y$ and the coupling term on the left and right side of the separation are given by

$$\begin{aligned} \alpha_L(x,\omega) &= \alpha(\omega)\,\theta(x-a/2) \\ \alpha_R(x,\omega) &= \alpha(\omega)\,\theta(-x-a/2)\,. \end{aligned} \tag{IV.136}$$

A slightly different approach to the path integral is to expand the field amplitudes in terms of their Fourier components, which due to the homogeneity in the directions perpendicular to the surface of the plates, would be for the scalar field

$$\phi(\mathbf{x},t) = \phi\left(x,\mathbf{x}_{||},t\right) = \int \frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\frac{d\omega}{2\pi}\phi\left(x,\mathbf{k}_{||},\omega\right)e^{i\left(\mathbf{k}_{||}\cdot\mathbf{x}_{||}-\omega t\right)}. \tag{IV.137}$$

In this case, the path integral is carried out in a very similar manner, where both intermediate path integrals must be evaluated first, followed by the main path integral over the scalar field ϕ. Since the the Fourier components are complex, care must be taken to to rewrite the exponents in terms of the real and imaginary parts of the Fourier components as in the discrete case (IV.46) before integrating. Including only the currents j for the scalar field, we eventually arrive at the result

$$Z[j] = Z^{-1}[0]\times \\ \exp\left(-\frac{i}{2\hbar}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \int\frac{d^2\mathbf{k}_{||}}{(2\pi)^2}\int\frac{d\omega}{2\pi}\left[j\left(x,\mathbf{k}_{||},\omega\right)G\left(x,x',\mathbf{k}_{||},\omega\right)j\left(x',-\mathbf{k}_{||},-\omega\right)\right]\right) \tag{IV.138}$$

where the Green's function now can be shown to satisfy

$$\left[\partial_x^2 - \mathbf{k}_{||}^2 + \frac{\omega^2}{c^2} + \frac{\omega^2}{c^2}\chi_L\left(x,\omega\right) + \frac{\left(\omega + vk_y\right)^2}{c^2}\chi_R\left(x,\omega + vk_y\right)\right] \\ \times G\left(x, x', \mathbf{k}_{||}, \omega\right) = \delta\left(x - x'\right). \tag{IV.139}$$

Even in this very simple case, we see the beginnings of the difficulties we will encounter if we try to solve the path integrals exactly, as we did for the stationary case. In fact, if instead of taking the motion parallel to the surfaces, we taken the motion to be perpendicular, i.e $\mathbf{v} = v\hat{\mathbf{z}}$, then we find the exact same results except that now the Green's function must satisfy

$$\left[\partial_x^2 - \mathbf{k}_{||}^{\mathbf{2}} + \frac{\omega^2}{c^2} + \omega^2\chi_R\left(x,\omega\right)\right] G\left(x, x', \mathbf{k}_{||}, \omega\right) + \int_{-\infty}^{\infty} dx''\left(i\omega - v_x\partial_{x''}\right)\left(-i\omega - v_x\partial_x\right) \times \\ \left[\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int_0^{\infty} d\omega' \frac{\alpha_L\left(x,\omega'\right)\alpha_L\left(x'',\omega'\right)}{\left(\omega - v_xk_x\right)^2 - \omega'^2} e^{ik_x\left(x - x''\right)}\right] G\left(x'', x', \mathbf{k}_{||}, \omega\right) = \delta\left(x - x'\right). \tag{IV.140}$$

Finding an exact solution to this equation is a daunting task. Simply applying the techniques used for the stationary case leads to difficulties even for the simplest of systems. Therefore the question becomes, how does one approach the path integrals over the field amplitude ϕ and the amplitude of the bath of oscillators X_ω, again the results of the last sections cannot be applied directly. It seems that in these, more complicated systems, we must turn to the perturbation methods that the path integral representation automatically provides.

Path Integrals: A Perturbative Approach

For the generic time-dependent systems with which we are concerned in this work, it seems that save for perhaps the occasional exceptional system, perturbation theory is likely to be the most practical approach to finding the correlation functions of the field amplitudes. We will now describe how to tackle such a problem using the real time path integral formulation of macroscopic QED.

Consider the general time dependent Lagrangian (IV.128) written in terms of the time-independent Lagrangian density $\mathscr{L}_0$ of [84] plus a time dependent part

$$\mathscr{L} = \mathscr{L}_0 + (\nabla \times \mathbf{A}) \cdot \int_0^\infty d\omega \, \boldsymbol{\alpha}_B(\omega, \mathbf{x}, t)\, \mathbf{X}_\omega - \dot{\mathbf{A}} \cdot \int_0^\infty d\omega \, \delta\alpha(\omega, \mathbf{x}, t)\, \mathbf{X}_\omega + \int_0^\infty d\omega \left[\mathbf{v}(\mathbf{x}, t) \cdot \nabla\right] \mathbf{X}_\omega \cdot \dot{\mathbf{X}}_\omega \tag{IV.141}$$

where the three terms are analogous to the three contributions to the time dependent Hamiltonian of section III.4. The generating functional may be defined in a similar manner to (IV.101) where the action is now written in terms of a time indepdenent part S_0 and a time dependent part S_δ as $S[\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega] = S_0[\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega] + S_\delta[\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega]$ where the time independent part is

$$S_0[\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega] = \int d^4x \; \mathscr{L}_0(\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega) \tag{IV.142}$$

and the time dependent part is given in terms of the extra contributions

$$S_\delta[\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega] = \int d^4x \int_0^\infty d\omega \left\{ (\nabla \times \mathbf{A}) \cdot \boldsymbol{\alpha}_B(\omega, \mathbf{x}, t) \cdot \mathbf{X}_\omega - \dot{\mathbf{A}} \cdot \delta\alpha(\omega, \mathbf{x}, t)\, \mathbf{X}_\omega + \left[\mathbf{v}(\mathbf{x}, t) \cdot \nabla\right] \mathbf{X}_\omega \cdot \dot{\mathbf{X}}_\omega \right\}. \tag{IV.143}$$

Now, using the properties of exponentials, we can write the time dependent part as factor to the free part, and expand it as a series

$$\exp\left\{ \frac{i}{\hbar} S_\delta[\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega] \right\} = \sum_{n=0}^\infty \frac{\left(\frac{i}{\hbar} S_\delta[\mathbf{A}, \mathbf{X}_\omega, \mathbf{Y}_\omega]\right)^n}{n!}. \tag{IV.144}$$

When the time dependent parts are small enough, it is only necessary to find the first few terms of this series. Written in this form, the full generating functional can be expressed as

$$Z[\mathbf{j}, \mathbf{z}_\omega, \mathbf{w}_\omega] = \sum_{n=0}^\infty Z_n[\mathbf{j}, \mathbf{z}_\omega, \mathbf{w}_\omega] \tag{IV.145}$$

where

$$Z_n\left[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega\right]=Z^{-1}\left[0,0,0\right]\left(\frac{i}{\hbar}\right)^n\frac{1}{n!}\int D\mathbf{A}\int D\mathbf{X}_\omega\int D\mathbf{Y}_\omega\left(S_\delta\left[\mathbf{A},\mathbf{X}_\omega,\mathbf{Y}_\omega\right]\right)^n$$
$$\times\exp\left\{\frac{i}{\hbar}S_0\left[\mathbf{A},\mathbf{X}_\omega,\mathbf{Y}_\omega\right]+\frac{i}{\hbar}\int d^3\mathbf{x}\int dt\left[\mathbf{A}\cdot\mathbf{j}+\int\limits_0^\infty d\omega\,\left(\mathbf{X}_\omega\cdot\mathbf{z}_\omega+\mathbf{Y}_\omega\cdot\mathbf{w}_\omega\right)\right]\right\}. \tag{IV.146}$$

To understand how to compute the terms of the series Z_n, consider that taking a derivative of the time independent generating functional (IV.101) with respect to the current $\mathbf{Z}_\omega$ for example, brings down a factor of $\mathbf{X}_\omega$ into the path integral. To build a factor of S_δ in the path integral, we have to take a more complicated combination of derivatives. Consider the example of the Z_1 term, in this case writing it in terms of combinations of derivatives gives

$$\begin{aligned}Z_1\left[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega\right]&=Z^{-1}\left[0,0,0\right]\\&\times\frac{i}{\hbar}\int d^4x\int\limits_0^\infty d\omega\left\{\left(\boldsymbol{\alpha}_B\left(\omega,\mathbf{x},t\right)\cdot\frac{\partial}{\partial\mathbf{z}_\omega\left(\mathbf{x},t\right)}\right)\cdot\left(\nabla\times\frac{\partial}{\partial\mathbf{j}\left(\mathbf{x},t\right)}\right)\right.\\&\qquad-\delta\alpha\left(\omega,\mathbf{x},t\right)\frac{\partial}{\partial\mathbf{z}_\omega\left(\mathbf{x},t\right)}\cdot\frac{\partial}{\partial t}\frac{\partial}{\partial\mathbf{j}\left(\mathbf{x},t\right)}\\&\qquad\left.-\frac{\partial}{\partial\mathbf{z}_\omega\left(\mathbf{x},t\right)}\cdot\left[\mathbf{v}\left(\mathbf{x},t\right)\cdot\nabla\right]\frac{\partial}{\partial t}\frac{\partial}{\partial\mathbf{z}_\omega\left(\mathbf{x},t\right)}\right\}Z_0\left[\mathbf{j},\mathbf{z}_\omega,\mathbf{w}_\omega\right].\end{aligned} \tag{IV.147}$$

Therefore, the only path integral one has to carry out here is the stationary, time-independent path integral of section IV.3, and all orders of perturbation theory can worked out by taking derivatives in a similar manner to the above. If we wish to compute correlation functions as in section III.4.6, we must then take the appropriate derivatives of the full generating functional composed of all the relevant orders of perturbation theory. For example, the two point function of the electromagnetic vector potential is found by taking the second order derivative with respect to the current $\mathbf{j}$ which to first order in perturbation theory would be

$$\begin{aligned}\langle 0|\,\hat{T}\left[\hat{\mathbf{A}}\left(\mathbf{x},t\right)\otimes\hat{\mathbf{A}}\left(\mathbf{x}',t'\right)\right]|0\rangle &= \left(\frac{\hbar}{i}\right)^{2}\left.\frac{\partial^{2}Z\left[\mathbf{j},\mathbf{z}_{\omega},\mathbf{w}_{\omega}\right]}{\partial\hat{\mathbf{j}}\left(\mathbf{x},t\right)\otimes\partial\hat{\mathbf{j}}\left(\mathbf{x}',t'\right)}\right|_{\mathbf{j}=\mathbf{z}_{\omega}=\mathbf{w}_{\omega}=0}\\ &\approx\left(\frac{\hbar}{i}\right)^{2}\left.\frac{\partial^{2}Z_{0}\left[\mathbf{j},\mathbf{z}_{\omega},\mathbf{w}_{\omega}\right]}{\partial\hat{\mathbf{j}}\left(\mathbf{x},t\right)\otimes\partial\hat{\mathbf{j}}\left(\mathbf{x}',t'\right)}\right|_{\mathbf{j}=\mathbf{z}_{\omega}=\mathbf{w}_{\omega}=0}\\ &+\left(\frac{\hbar}{i}\right)^{2}\left.\frac{\partial^{2}Z_{1}\left[\mathbf{j},\mathbf{z}_{\omega},\mathbf{w}_{\omega}\right]}{\partial\hat{\mathbf{j}}\left(\mathbf{x},t\right)\otimes\partial\hat{\mathbf{j}}\left(\mathbf{x}',t'\right)}\right|_{\mathbf{j}=\mathbf{z}_{\omega}=\mathbf{w}_{\omega}=0}.\end{aligned} \tag{IV.148}$$

In order to simplify the computation of these terms, after considering the form of the generating functional Z_0 of section IV.3, it is worth providing some useful functional derivative identities that can be worked out using the definition (IV.96). For example, for a functional F defined by

$$F\left[f\left(x\right)\right]=\int dx\int dx'\left[g\left(x\right)+f\left(x\right)\right]A\left(x,x'\right)\left[f\left(x'\right)+g\left(x'\right)\right] \tag{IV.149}$$

then we find the identity

$$\frac{\delta F\left[f\left(x\right)\right]}{\delta f\left(y\right)}=\int dx\left[g\left(x\right)+f\left(x\right)\right]\left[A\left(x,y\right)+A\left(y,x\right)\right]. \tag{IV.150}$$

Similarly, for a slightly more complicated functional defined by

$$\begin{aligned}F\left[h_{\omega}\left(t\right)\right]=\int dt\int dt'&\left[\int d\omega'\int dt''f_{\omega'}\left(t-t''\right)h_{\omega'}\left(t''\right)+g\left(t\right)\right]A\left(t,t'\right)\\ &\times\left[\int d\omega'\int dt'''f_{\omega'}\left(t'-t'''\right)h_{\omega'}\left(t'''\right)+g\left(t'\right)\right]\end{aligned} \tag{IV.151}$$

the functional derivative with respect to h_ω is given as

$$\begin{aligned}\frac{\delta F\left[h_{\omega}\left(t\right)\right]}{\delta h_{\omega_{1}}\left(t_{1}\right)}=\int dt\int dt'&\left[\int d\omega'\int dt''f_{\omega'}\left(t-t''\right)h_{\omega'}\left(t''\right)+g\left(t\right)\right]\\ &\times f_{\omega_{1}}\left(t'-t_{1}\right)\left[A\left(t,t'\right)+A\left(t',t\right)\right].\end{aligned} \tag{IV.152}$$

Furthermore, once (IV.147) has been computed using the above derivative identities, we then need to take the second derivative with respect to the currents $\mathbf{j}$. We can make this easier by noting that Z_1 will take the form $Z_1 = A\, e^I$, where both A

and I are functionals of the currents. Then the second derivative with respect to the current can be written as

$$\frac{\partial^2 Z_1}{\partial f(y)\,\partial f(z)} = \left(\frac{\partial^2 A}{\partial f(y)\,\partial f(z)} + \frac{\partial A}{\partial f(y)}\,\frac{\partial I}{\partial f(z)}\right) e^I \\ + \left(\frac{\partial A}{\partial f(z)}\frac{\partial I}{\partial f(y)} + A\,\frac{\partial^2 I}{\partial f(y)\,\partial f(z)} + A\,\frac{\partial I}{\partial f(y)}\frac{\partial I}{\partial f(z)}\right) e^I. \tag{IV.153}$$

But since I is quadratic in the current, some terms will disappear once the currents are set to zero, and this is just equivalent to

$$\frac{\partial^2 Z_1}{\partial f(y)\,\partial f(z)} \equiv \left[\frac{\partial^2 A}{\partial f(y)\,\partial f(z)} + A\frac{\partial^2 I}{\partial f(y)\,\partial f(z)}\right] Z_0 \tag{IV.154}$$

Combining the identities (IV.150),(IV.152) and (IV.154) allows one to compute the first few orders of perturbation theory from which one may establish a set of rules for computing higher order terms. We will not do this here but instead leave this for future work.

More pressing is the fact that while these time ordered correlation functions are indeed useful, this real time formulation does not allow us to directly compute the thermal correlation functions of Section III.4.6, used to describe the fluctuation-dissipation theorem. In order to achieve this, we require a formulation that allows us to refer to the real time parameter describing the time evolution of the time-dependent system while using the imaginary time formulation that mimics the temperature dependence. In other words, we require a formulation that lends itself to both real and imaginary time.

IV.4 Schwinger-Keldysh Formalism

It is our goal to ascertain the thermal two point functions of the electromagnetic field near time-dependent or moving media as described by the macroscopic QED model of Chapter III. The vacuum expectations of the field amplitudes are found when the low temperature limit ($T \to 0$) is taken.

Except for the simple case of constant motion in very simple geometries, quantisation of such a system is unfeasible as is a direct application of non-perturbative path integral methods. Canonical perturbation theory provides a root but for higher order terms results in very lengthy calculations whereas the path integral approach facilitates a somewhat faster route to establishing a set of rules from which higher order terms can be found.

In the previous chapter, we showed how the path integral methods can be applied to macro QED for real-time or imaginary-time correlation functions to retrieve either the Feynman correlation functions or the equal time correlations functions for the time-independent systems. However, since the $T = 0$, vacuum correlation functions are in fact the $T \to 0$ limit of the thermal correlators of section IV.2.2, for time-dependent systems, where the real-time perturbation methods applied in section IV.3 are perfectly valid, it seems difficult to build such correlators as they will contain references to both real and imaginary time, something that we have not yet covered.

In this chapter, we explore a formalism that allows the explicit calculation of the correlation functions of the various field amplitudes for systems that are slowly time-varying and in thermal equilibrium by doing exactly this: combining the real and imaginary time formalisms. In essence, for the electric field for example, we will be calculating correlators of the form

$$\left\langle \hat{\mathbf{E}}(\mathbf{x}, t) \otimes \hat{\mathbf{E}}(\mathbf{x}', t') \right\rangle_T = \mathrm{Tr}\left[\mathrm{e}^{-\beta \hat{H}}\, \hat{\mathbf{E}}(\mathbf{x}, t) \otimes \hat{\mathbf{E}}(\mathbf{x}', t') \right]. \tag{IV.155}$$

Note that unlike in section III.4.6, the times can be *different* here. Setting these two *real* time parameters to be equal $t = t'$ retrieves the equal time correlation functions of section III.4.6.

The formalism we will attempt to apply to macroscopic QED is referred to as the Schwinger-Keldysh path integral formalism (see [50, 101, 102] and references therein). The basic concept involves calculating path integrals in both real *and* imaginary times. Since, macroscopic QED is essentially a system composed of an infinite set of harmonic oscillators, we shall now illustrate this idea more rigorously with the simplest case of just one of these harmonic oscillators.

Simple Harmonic Oscillator

The macroscopic QED model is build upon the interaction of an infinite number of interacting harmonic oscillators. As such, in order to understand how to apply this non-equilibrium method to macroscopic QED, we must first understand how to apply it to one harmonic oscillator.

The thermal expectation of the amplitude X_ω of the harmonic oscillator of natural frequency ω at a time t is found similarly to (IV.155) via the trace

$$\left\langle \hat{\mathsf{X}}_\omega(t) \right\rangle_T = \mathsf{Tr}\left[\mathrm{e}^{-\beta \hat{H}}\, \hat{\mathsf{X}}_\omega(t)\right] \tag{IV.156}$$

where the operator $X(t)$ is the time dependent operator of the Heisenberg picture (see section I.3). The time dependence can be factored out using the unitary evolution operator $\hat{U}(t_1, t_2) = \hat{T} \exp\left(\int_{t_1}^{t_2} dt\, \hat{H}(t)\right)$, from which the expectation of the amplitude operator can be written as

$$\left\langle \hat{\mathsf{X}}_\omega(t) \right\rangle_T = \mathsf{Tr}\left[\mathrm{e}^{-\beta \hat{H}} \hat{U}(t_0, t)\, \hat{\mathsf{X}}_\omega\, \hat{U}(t, t_0)\right] \tag{IV.157}$$

where t_0 is the time at which the operators in both the Schroedinger and the Heisenberg pictures coincide. According to the Schwinger-Keldysh formalism [50, 101, 102], each exponent can be reinterpreted as a section of a contour in complex time. The thermal part as an integral over purely imaginary time, as in Section IV.2.2, and the evolution operators as real time parts as in Section IV.3, with the full complex time path given in Figure IV.5 a). In the limit when $t_0 \to -\infty$, the path then becomes that of IV.5 b). Such correlation functions can be derived from the generating functional

$$Z\left[h_\omega\right] = \int D\mathsf{X}_\omega\, \exp\left(\frac{i}{\hbar} S_c\left[\mathsf{X}_\omega\right] + \frac{i}{\hbar} \oint d\tau\, \mathsf{X}_\omega \cdot h_\omega\right) \tag{IV.158}$$

where the "contour action" S_c is defined similarly to the usual action except that the time integral is over the time contour $S_c[\mathsf{X}_\omega] = \oint d\tau\, L[\mathsf{X}_\omega, \partial_\tau \mathsf{X}_\omega]$ where the contour is described in Figure IV.5.

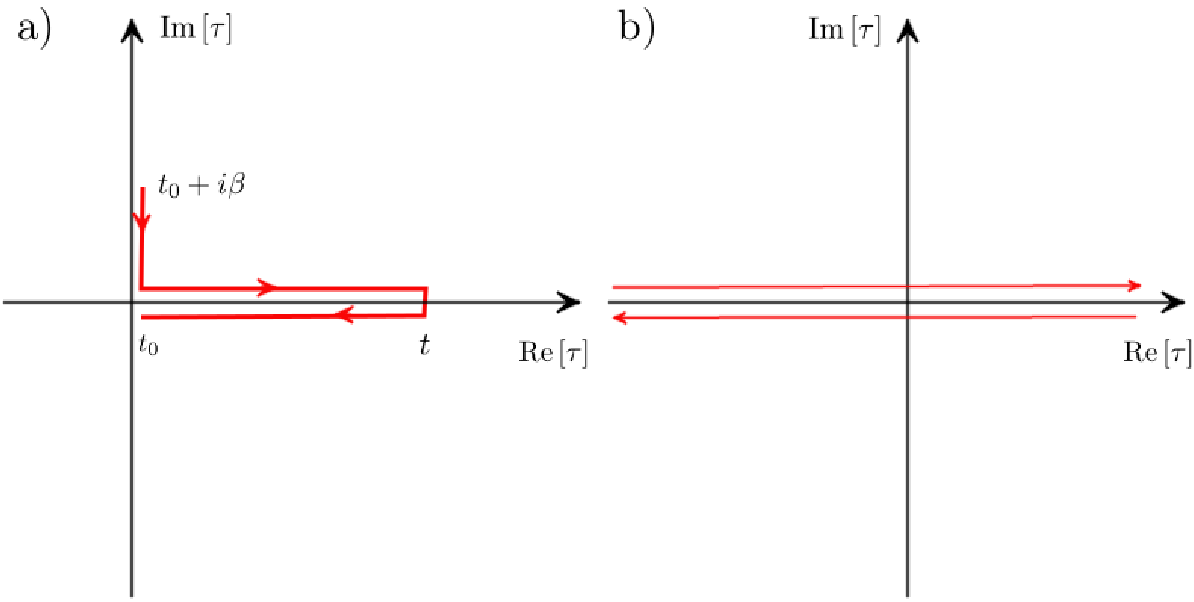

Figure IV.5: a) The complex time contour in the Schwinger-Keldysh path integral begins by a purely imaginary part arising from the $\exp(-\beta\hat{H})$ as in the Matsubara imaginary time formalism, then runs slightly above the real time axis from t_0 to t and then back down the real time axis slightly underneath it to t_0. b) When the interaction Hamiltonian is considered to be active for long times, the thermal part decouples from the rest of the exponent and cancels out with a normalisation [103], leaving the contour depicted which runs from $-\infty$ to ∞ ubove the real time axis and then back again underneath the real time axis to form a closed loop in complex time. The thermal properties are then guaranteed by the KMS condition on the fields (IV.164).

The path integral within the generating functional can be computed explicitly in a similar manner to previously via the change of variables

$$\mathsf{X}_\omega(\tau) \to \mathsf{X}_\omega(\tau) + \oint d\tau' D_c^{-1}(\tau, \tau')\, h_\omega(\tau') \tag{IV.159}$$

where the matrix $D_c(\tau - \tau')$ is defined as

$$D_c(\tau - \tau') = \left(\partial_{\tau'}^2 + \omega^2\right) \delta(\tau - \tau'). \tag{IV.160}$$

The generating functional is found as before and normalised such that $Z = 1$

when the currents are set to zero,

$$Z\left[h_\omega\right] = Z^{-1}\left[0\right]\exp\left(\frac{i}{\hbar}\oint d\tau \oint d\tau'\; h_\omega(\tau)\cdot G_c(\tau-\tau')\cdot h_\omega(\tau')\right) \qquad \text{(IV.161)}$$

where the Green's function on the contour $G_c(\tau-\tau') = D_c^{-1}\left(\tau,\tau'\right)$ must satisfy the integral expression

$$\oint d\tau' D_c\left(\tau,\tau'\right) D_c^{-1}\left(\tau,\tau'\right) = \delta_c\left(\tau-\tau'\right) \qquad \text{(IV.162)}$$

which, after evaluation is equivalent to the differential equation

$$\left(\partial_\tau^2+\omega^2\right) G_\omega^c\left(\tau-\tau'\right) = \delta_c\left(\tau-\tau'\right) \qquad \text{(IV.163)}$$

with the KMS boundary conditions

$$G_\omega^c\left(\tau-i\beta+\tau'\right) = G_\omega^c\left(\tau-\tau'\right). \qquad \text{(IV.164)}$$

Customarily, if the complex time parameter τ is on the upper line, the variable is given the subscript "$+$", and if it is on the lower then it is given the subscript "$-$". Integrals over the time path can then be written as the sum of two integrals $\oint d\tau = \int_{-\infty}^{\infty} dt_+ + \int_{\infty}^{-\infty} dt_-$. Thus the integral expression in the exponent of the generating functional (IV.161) can be rewritten as

$$\begin{aligned}
&\oint d\tau \oint d\tau'\; h_\omega(\tau)\, G_\omega^c\left(\tau-\tau'\right) h_\omega(\tau') = \\
&\int\limits_{-\infty}^{\infty} dt_+ \int\limits_{-\infty}^{\infty} dt'_+\, h_\omega(t_+)\, G_\omega^c\left(t_+-t'_+\right) h_\omega(t'_+) \\
&-\int\limits_{-\infty}^{\infty} dt_+ \int\limits_{-\infty}^{\infty} dt'_-\, h_\omega(t_+)\, G_\omega^c\left(t_+-t'_-\right) h_\omega(t'_-) \\
&-\int\limits_{-\infty}^{\infty} dt_- \int\limits_{-\infty}^{\infty} dt'_+\, h_\omega(t_-)\, G_\omega^c\left(t_--t'_+\right) h_\omega(t'_+) \\
&+\int\limits_{-\infty}^{\infty} dt_- \int\limits_{-\infty}^{\infty} dt'_-\, h_\omega(t_-)\, G_\omega^c\left(t_--t'_-\right) h_\omega(t'_-).
\end{aligned} \qquad \text{(IV.165)}$$

Using the change of notation $G^c_\omega\left(t_\pm - t'_\mp\right) \to G^{\pm\mp}_\omega(t-t')$, and $h_\omega(t_\pm) \to h^\pm_\omega(t)$, in a similar fashion to [50], we can rewrite this an integral over the matrix multiplication

$$\oint d\tau \oint d\tau'\ h_\omega(\tau)\, G^c_\omega(\tau-\tau')\, h_\omega(\tau') \equiv \int\limits_{-\infty}^{\infty} dt \int\limits_{-\infty}^{\infty} dt'\, \mathbf{h}_\omega(t) \cdot \mathbf{G}_\omega(t-t') \cdot \mathbf{h}_\omega(t') \tag{IV.166}$$

where the current vector is defined as

$$\mathbf{h}_\omega(t) = \begin{pmatrix} h^+_\omega(t) \\ -h^-_\omega(t) \end{pmatrix} \tag{IV.167}$$

and the matrix of Green's functions via

$$\begin{aligned} \mathbf{G}_\omega(t-t') &\equiv \begin{pmatrix} G^{++}_\omega(t-t') & G^{+-}_\omega(t-t') \\ G^{-+}_\omega(t-t') & G^{--}_\omega(t-t') \end{pmatrix} \\ &\equiv \begin{pmatrix} G^{T}_\omega(t-t') & G^{<}_\omega(t-t') \\ G^{>}_\omega(t-t') & G^{\bar{T}}_\omega(t-t') \end{pmatrix} \end{aligned} \tag{IV.168}$$

where $G^{</>}_\omega$ are known as the lesser and greater Green's functions respectively. Using the boundary conditions (IV.164), and the fact that for both of these, both time parameters t and t' are on different branches of the contour, they are found to be

$$G^<_\omega(t-t') = -\frac{i}{\hbar}\left\langle \hat{\mathsf{X}}_\omega(t)\hat{\mathsf{X}}_\omega(t')\right\rangle = -\frac{i}{2\omega}\left\{[1+N(\omega)]\,\mathrm{e}^{\mathrm{i}\omega(t-t')} + N(\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}\right\} \tag{IV.169}$$

and

$$G^>_\omega(t-t') = -\frac{i}{\hbar}\left\langle \hat{\mathsf{X}}_\omega(t')\hat{\mathsf{X}}_\omega(t)\right\rangle = -\frac{i}{2\omega}\left\{N(\omega)\mathrm{e}^{\mathrm{i}\omega(t-t')} + [1+N(\omega)]\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\right\} \tag{IV.170}$$

The two remaining Green's functions $G^{T/\bar{T}}_\omega$ are the time ordered or time anti-ordered Green's functions found in the real time formalism. The time ordered Green's function corresponds to the case where both time parameters are on the upper branch, and the anti-time ordered when they are both on the lower branch. Written as expectation of operators and using the operator expressions (I.31), we can recover

$$G_{\omega}^{T}(t-t') = -\frac{i}{\hbar}\left\langle \hat{T}\big[\hat{\mathsf{X}}_{\omega}(t)\hat{\mathsf{X}}_{\omega}(t')\big]\right\rangle = \theta(t-t')G_{\omega}^{>}(t-t') + \theta(t'-t)G_{\omega}^{<}(t-t') \tag{IV.171}$$

and

$$G_{\omega}^{\bar{T}}(t-t') = -\frac{i}{\hbar}\left\langle \hat{\bar{T}}\big[\hat{\mathsf{X}}_{\omega}(t)\hat{\mathsf{X}}_{\omega}(t')\big]\right\rangle = \theta(t-t')G_{\omega}^{<}(t-t') + \theta(t'-t)G_{\omega}^{>}(t-t')\,. \tag{IV.172}$$

respectively. Note that there is a more useful way of expressing the exponent of (IV.161) that uses a rotation first used by Keldysh [104]. The exponent in (IV.161) is rewritten as

$$\int\limits_{-\infty}^{\infty} dt \int\limits_{-\infty}^{\infty} dt'\, \tilde{\mathbf{h}}_{\omega}(t)\cdot\tilde{\mathbf{G}}_{\omega}(t-t')\cdot\tilde{\mathbf{h}}_{\omega}(t') \tag{IV.173}$$

by defining a new current vector

$$\tilde{\mathbf{h}}_{\omega}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\cdot\mathbf{h}_{\omega}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}h_{\omega}^{+}(t)+h_{\omega}^{-}(t)\\ h_{\omega}^{+}(t)-h_{\omega}^{-}(t)\end{pmatrix} \tag{IV.174}$$

and a similarly a new Green's function matrix

$$\begin{aligned}\tilde{\mathbf{G}}_{\omega}(t-t') &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\cdot\mathbf{G}_{\omega}(t-t')\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\\ &= \begin{pmatrix}0 & G_{\omega}^{A}(t-t')\\ G_{\omega}^{R}(t-t') & G_{\omega}^{K}(t-t')\end{pmatrix}\end{aligned} \tag{IV.175}$$

where $G_{\omega}^{R/A}$ are the retarded and advanced Green's function encountered in the real time formalism and G_{ω}^{K} is the Keldysh Green's function. They are defined in terms of the previous four components via the relations

$$\begin{aligned} G^R_\omega(t-t') &= \frac{1}{2}\left[G^T_\omega(t-t') + G^>_\omega(t-t') - G^<_\omega(t-t') - G^{\bar{T}}_\omega(t-t')\right] \\ G^A_\omega(t-t') &= \frac{1}{2}\left[G^T_\omega(t-t') - G^>_\omega(t-t') + G^<_\omega(t-t') - G^{\bar{T}}_\omega(t-t')\right] \\ G^K_\omega(t-t') &= \frac{1}{2}\left[G^T_\omega(t-t') + G^>_\omega(t-t') + G^<_\omega(t-t') + G^{\bar{T}}_\omega(t-t')\right]. \end{aligned} \tag{IV.176}$$

Here we immediately see the use in performing this rotation, if one is looking for the retarded Green's function for example, one need only take a derivative with respect to these new currents. In short, the Schwinger Keldysh formalism provides a self-contained method for dealing with thermal systems out of equilibrium. This means that one could have, for example, time-dependent variations in position and momentum of the constituent particles as well time and space dependent changes in temperature. Ultimately we wish to understand and model how the quantum vacuum changes in the vicinity of media with arbitrary time-dependencies. The vacuum correlators that would allow such an understanding our found by taking the zero temperature limit of the thermal correlators. Thus, if we can apply this method to the macroscopic QED action, we will have the appropriate framework to explore these correlators using the path integral method for perturbation theory as well as a model capable of exploring more complex, thermal systems. Therefore, let us now begin the process of applying this model to macroscopic QED.

Keldysh Formalism for Macroscopic QED

For macroscopic QED, the situation should be very similar, we therefore begin by proceeding analogously to sections IV.2.2 and IV.3. We will again consider the simpler, scalar field case we use in section IV.3.2. This time, we must begin from the contour path integral defined by the contour action, which is the contour time integral over the stationary scalar Lagrangian density

$$S_c[\phi, X_\omega] = \int_{-\infty}^{\infty} dx \oint d\tau\, \mathscr{L}_0[\phi, X_\omega] \tag{IV.177}$$

where the Lagrangian density is given by (IV.117) and where the time parameter is promoted to the complex contour time parameter τ. If we follow the same procedures as before, using the properties of contour integrals and derivatives outlined in the harmonic oscillator example, after setting $h_\omega \to 0$ for simplicity, we arrive at an analogous generating functional

$$Z[j,0] = Z[0,0] \exp\left(\frac{i}{2\hbar} \int_{-\infty}^{\infty} dx \oint d\tau \oint d\tau' \, j(\tau) \, A^{-1}(x,x',\tau,\tau') \, j(\tau')\right) \quad \text{(IV.178)}$$

where we have set the oscillator currents to zero here for simplicity. The condition on the A matrix now becomes

$$\int_{-\infty}^{\infty} dx'' \oint d\tau'' \, A(x,x'',\tau,\tau'') \, A^{-1}(x'',x',\tau'',\tau') = \delta(x-x') \, \delta_c(\tau-\tau') \, . \quad \text{(IV.179)}$$

Performing the integral over position and the contour integral over complex time, leaves the integro-differential equation for the macroscopic Green's function in complex time

$$\begin{aligned} &\left(\partial_x^2 - \frac{1}{c^2}\partial_\tau^2\right) G(x,x',\tau,\tau') \\ &+ \int_0^\infty d\omega \, \alpha^2(x,\omega) \oint d\tau'' \, \partial_\tau \partial_{\tau''} \, G_\omega^c(\tau-\tau'') \, G(x,x',\tau'',\tau') = \delta(x-x') \, \delta_c(\tau-\tau') \, . \end{aligned} \quad \text{(IV.180)}$$

This is a little more complex than the real time or purely imaginary time equations previously encountered. To make sense of it, we can use the same trick as for the harmonic oscillator and rewrite this equation in terms of the matrix of Greens functions

$$\begin{aligned} &\left(\partial_x^2 - \frac{1}{c^2}\partial_t^2\right) \mathbf{G}(x,x',t,t') \\ &+ \int_0^\infty d\omega \, \alpha^2(x,\omega) \int_{-\infty}^{\infty} dt'' \, \partial_t \partial_{t''} \mathbf{G}_\omega(t-t'') \cdot \mathbf{G}(x,x',t'',t') = \delta(x-x') \, \delta(t-t') \end{aligned} \quad \text{(IV.181)}$$

where $\mathbf{G}_\omega\left(t-t''\right)$ is defined according to (IV.168) and the full Green's function matrix is similarly defined by

$$\mathbf{G}\left(x,x',t,t'\right)=\begin{pmatrix} G^T\left(x,x',t,t'\right) & G^<\left(x,x',t,t'\right) \\ G^>\left(x,x',t,t'\right) & G^{\bar{T}}\left(x,x',t,t'\right). \end{pmatrix} \tag{IV.182}$$

The reader will notice that this is, however, a set of four coupled equations. We can perform the Keldysh rotation according (IV.175) in order to obtain a set of equations for the retarded, advanced and Keldysh Green's functions that appear in the real and imaginary time formalisms already discussed. However, unfortunately the resulting four equations are also coupled. As they are, they don't seem to retrieve the equations that we expect from the simpler cases derived above.

In theory, since the system is comprised of coupled harmonic oscillators, it should be possible to extend this method to macroscopic QED. However, it seems this will be more subtle than using the methods developed for the purely real or purely imaginary time formalisms applied to the method described for one harmonic oscillator. One way forward might be to first of all consider two coupled harmonic oscillators and understand how their dynamics unfolds using this method.

For example, if the two oscillators were coupled linearly with some coupling constant κ then the generating functional could be written in terms of the path integral

$$Z\left[j_X,j_Y\right]=\int DX\int DY\,\exp\left[\frac{i}{\hbar}\oint d\tau\,\left(L_{XY}+j_X\,X+j_Y\,Y\right)\right] \tag{IV.183}$$

where the currents J_X and J_Y are the currents for each of the two oscillators with amplitudes X and Y. The Lagrangian could be given by

$$L_{XY}=\frac{1}{2}\left(\dot{X}^2-\omega^2X^2\right)+\frac{1}{2}\left(\dot{Y}^2-\omega^2Y^2\right)+\kappa XY \tag{IV.184}$$

where we have taken the two natural frequencies to be identical. The advantage of considering this case is that performing the transformations

$$
\begin{aligned}
X &\to \frac{1}{\sqrt{2}}(A+B) \\
Y &\to \frac{1}{\sqrt{2}}(A-B)
\end{aligned}
\tag{IV.185}
$$

would lead to the new, uncoupled Lagrangian L_{AB}

$$
L_{AB} = \frac{1}{2}\left(\dot{A}^2 - \omega_A^2 A^2\right) + \frac{1}{2}\left(\dot{B}^2 - \omega_B^2 B^2\right) \tag{IV.186}
$$

where the modified natural frequencies of the oscillators satisfy

$$
\begin{aligned}
\omega_A^2 &= \omega^2 - \frac{\kappa}{2} \\
\omega_B^2 &= \omega^2 + \frac{\kappa}{2}.
\end{aligned}
\tag{IV.187}
$$

This in turn, would lead to a new expression for the generating functional

$$
Z\left[j_A, j_B\right] = \int DA \int DB \, \exp\left[\frac{i}{\hbar}\oint d\tau \, \left(L_{AB} + j_A \, A + j_B \, B\right)\right] \tag{IV.188}
$$

in terms of the new currents

$$
\begin{aligned}
j_A &= \frac{1}{\sqrt{2}}\left(j_X + j_Y\right) \\
j_B &= \frac{1}{\sqrt{2}}\left(j_X - j_Y\right).
\end{aligned}
\tag{IV.189}
$$

Therefore on the one hand, we have two uncoupled harmonic oscillators A and B whose contour Green's functions will be those previously discussed, but on the other hand, the coupled case exhibits a similar difficulty to that of our first attempt at applying the Schwinger-Keldysh formalism to macroscopic QED above. One can see this by explicitly performing the path integral. If we choose to integrate out the Y oscillator first and then the X oscillator (or vice-versa, but the system is symmetric) then we would arrive at the expression for the generating functional

$$Z\left[j_X,\, j_Y\right] = Z^{-1}\left[0,0\right]\, \exp\left[\frac{i}{2\hbar}\oint d\tau\oint d\tau'\times\right.$$
$$\Bigg(\left\{j_X\left(\tau\right)+\frac{\kappa}{2}\oint d\tau'' j_Y\left(\tau\right)\left[G_Y\left(\tau-\tau''\right)+G_Y\left(\tau''-\tau\right)\right]\right\}$$
$$\cdot G_X\left(\tau-\tau'\right)\cdot\left\{j_X\left(\tau'\right)+\frac{\kappa}{2}\oint d\tau'' j_Y\left(\tau'\right)\left[G_Y\left(\tau''-\tau'\right)+G_Y\left(\tau'-\tau''\right)\right]\right\}$$
$$\left. +\, j_Y\left(\tau\right)G_Y\left(\tau-\tau'\right)j_Y\left(\tau'\right)\Bigg)\right] \tag{IV.190}$$

where again, after setting the second current to zero, in this case j_Y, we arrive at the simpler analogue of (IV.180)

$$Z\left[j_X,0\right] = Z^{-1}\left[0,0\right]\,\exp\left[\frac{i}{2\hbar}\oint d\tau\oint d\tau'\, j_X\left(\tau\right)\cdot G_X\left(\tau-\tau'\right)\cdot j_X\left(\tau'\right)\right] \tag{IV.191}$$

where the Green's function $G_X\left(\tau-\tau'\right)$ satisfies the equation

$$\left[\partial_\tau^2+\omega^2-\kappa^2 G_Y\left(\tau-\tau'\right)\right]G_X\left(\tau-\tau'\right)=\delta_c\left(\tau-\tau'\right) \tag{IV.192}$$

which is of course, analogous to (IV.181). The Green's function $G_Y\left(\tau-\tau'\right)$ appearing in (IV.192) satisfies

$$\left(\partial_\tau^2+\omega^2\right)G_Y\left(\tau-\tau'\right)=\delta_c\left(\tau-\tau'\right). \tag{IV.193}$$

Writing (IV.192) in terms of the matrix of Green's function brings us to the same problem encountered above.

This two body system has the advantage of its simplicity and may lead to an understanding that will allow us to complete the adaptation of macroscopic QED to this method. We leave the completion of this task for future work.

Chapter V

Conclusion

V.1 Summary

Throughout this work, it has been our aim to contribute to the understanding of the quantum vacuum in the vicinity of non-uniformly moving, or more generally, time-dependent magneto-dielectrics.

We saw that the quantum vacuum, as described by quantum field theory, is a fluctuating field of infinite energy. It has real physical effects and as such, understanding how the behaviour of magneto-dielectrics changes this vacuum is useful in order to quantify the influence of the vacuum on said bodies. Such an understanding has many uses where the forces generated by the quantum vacuum become important [73].

Near moving media represented by a macroscopic constant real refractive index, one effect of the motion on the form of the vacuum is the appearance of negative frequency modes of light, absent when this medium is stationary. Their existence suggests a deeper effect but their interpretation remained obscure until now. In order to demystify these modes, we considered the same problem but using a description of the medium containing a simple approximation to the degrees of freedom of the constituent particles and showed that, microscopically, such modes no longer exist but that they emerge after spatial averaging over the microscopic structure. At the microscopic level, the point at which the frequencies in the macroscopic limit become negative corresponds to a threshold speed of the scatterers.

Furthermore, we showed that when an oscillator is placed within this medium, the electromagnetic field serves to both amplify and damp the motion of the oscillator. However, if the speed is larger than the speed of light in the medium, the radiation damping is reversed leading to an exchange of energy between the moving medium and the oscillator.

Another effect of the motion on the vacuum was the idea the vacuum field could be 'dragged' along with the medium. We analysed the Poynting vector of the electromagnetic field outside a uniformly moving dielectric and showed that while such modes did appear, spatial symmetries meant that they can effectively be removed rather arbitrarily from the Poynting vector. This is not paradoxical and stems from the very definition of the energy-momentum tensor. This is a subtle issue that does not necessarily negate the interpretation that such modes of light appear alongside moving media but rather implies that the Poynting vector is not best suited to its demonstration in spatially symmetric scenarios. When a dipole probe is placed in the vicinity of the medium, the symmetry is broken and an energy transfer between the two bodies is finite. In this case, one would suppose that it should not be possible to remove the Poynting vector as before. Although we did not calculate this explicitly in this work, it remains an interesting case to study in the future.

These first two cases contained no dispersion or dissipation of the electromagnetic energy. Real materials of course always exhibit these effects and only over restricted frequency ranges does one ever encounter materials with approximately real refractive indices. For many systems, such an approximation is perfectly valid. However, for many of the applications that concern the interaction of moving dielectrics within the quantum vacuum, the effects of dissipation and dispersion play vital roles and can no longer be ignored.

Using a fully quantised model that includes both effects for any magneto-dielectric whose electromagnetic response can be approximated by a frequency dependant permittivity and permeability, we considered how non-uniform motion and time-dependent permittivities of such magneto-dielectrics change the quantum vacuum. In this model, the system is described in terms of the polariton quasi-particles representing mixed excitations of the electromagnetic field and the degrees of freedom of the material.

To first order, pairs of polaritons are created out of the vacuum field due to the time dependence. It was found that their rate of creation depends heavily on the type of time-dependence as one would expect, but also on the resonances of the material. At the time of submission of this work, this was a new effect as previous cases did not include any dispersion.

The creation of polariton pairs was found to modify the vacuum correlation functions between the field amplitudes. We showed this by considering the first order change to the equal time two point functions that illustrate the fluctuation-dissipation theorem. An example of the use of these corrections to the correlators could be to approximate the changes in the energy stress tensor between moving bodies.

For higher order perturbative terms, the canonical formulation of quantum field theory is rather cumbersome and we therefore began to explore the path integral formulation of macroscopic QED. The ultimate goal is to use the extensive perturbative techniques of this method to establish a framework within which to calculate high order terms. We demonstrated the usual purely real time formulation of the path integral applied to macroscopic QED recovering the Feynman propagators of the theory, as well as the purely imaginary time (Matsubara) formulation allowing for the calculation of the equal time thermal correlation functions. The vacuum correlators of interest in this work are found by taking the zero temperature limit of the thermal correlators.

V.2 Future Work

In Section II.2, we discussed the origin of negative frequencies within moving media. To analyse this effect in the quantum regime, we considered the effect of this motion on the excitation of a two level system. However, the classical picture developed in this section suggests energy from the motion itself is extracted by the oscillator, analogously to quantum friction. As long as one supplies enough energy to maintain the motion, an unlimited amount of energy can be drawn from the material. This is, of course, not surprising. To test this in the quantum regime, one could try coupling more intricate systems to the field, starting with, for example, a three level system. If our picture is correct, one should be able obtain higher level excitations in the coupled system.

In Section II.3, we explored the idea that the quantum vacuum is ‘dragged’ along by moving objects. This is similar in concept to the old ether theories of 19th century electromagnetism. However, as we saw, the Lorentz invariance of the vacuum prohibits such effects unless the symmetry of the system is broken. One possible line of inquiry could therefore be to take the model used in this section and couple to it a two level system and see whether this symmetry breaking is sufficient for the energy transfer to occur. As far as the author knows, it is not currently known whether this energy exchange occurs because of the symmetry breaking or because of the inclusion of dissipation and dispersion of light within the moving medium, something that was not included in this chapter.

The final two Chapters made use of the Macroscopic QED models for the interaction of the electromagnetic field with magneto-dielectrics. In the final Chapter, we found that, for time dependent systems, the Matsubara formalism is no longer applicable. The Keldysh Schwinger formalism provides a route to marrying the real time approach and the imaginary time approach, allowing one to establish a perturbative series for time dependent systems. We began the application of this approach to macroscopic QED and showed the basic idea. The remainder of this task will be left for future work.

After adapting macroscopic QED to the Schwinger-Keldysh formalism, one will then be in a position to fully analyse the perturbation series and derive a set of rules from which to consider higher order terms. The utility of this will be in establishing to high accuracy the effect of any time-dependence of magneto-dielectrics where the time-dependence can be treated as a small change to an otherwise time-independent system.

Our findings and the possible future investigations may be very briefly summarised in the following table:

Findings	Implications and Future Work
A harmonic oscillator moving within a medium of refractive index n is both damped and amplified by the EM field if its velocity $v < c/$n but is only amplified when $v > c/$n. Quantum mechanically this corresponds to a finite excitation rate of the system only when $v > c/$n.	Novel description of energy transfer after a change of sign of the frequency in moving media that could help to understand similar effects (e.g. Hawking radiation). Future investigations: quantum mechanical excitation rates for more complex systems (3 levels or more).
Negative frequencies were found to be an artefact of macroscopic electromagnetism. Microscopically, such modes are no longer present. The change of sign of the frequency in the macroscopic regime corresponds to surpassing an interaction frequency threshold in the microscopic regime.	The microscopic picture may help us understand further the mechanisms for energy transfer between the moving bodies, perhaps shedding some light on quantum friction. One could investigate a similar model where the relative velocity is allowed to change. What about similar effects such as Hawking radiation? Is the origin of negative frequencies similar? Does this imply a stochastic structure of spacetime?

The intuitive picture of the field being 'dragged' along with the moving body remains unclear. The Poynting vector was shown to not necessarily be a useful tool in translationally invariant systems.	Investigate a similar model coupled to an oscillator to see whether the lack of symmetry leads to a Poynting that demonstrates this 'dragging' of the field. Investigate whether the inclusion of dissipation alone is sufficient to see this effect.
The polariton creation from the time dependence of the dielectric depends heavily on the absorption and dissipation of electromagnetic energy. Such changes in the field modify the correlation functions relevant to the fluctuation dissipation theorem.	This is a novel description taking into account dispersion and dissipation for time-dependent systems. Usually these effects are ignored. One could explore the effect of these modifications to results where this effect was not previously accounted for.
The possibility of using the path integral methodology for macroscopic QED was explored. The real time and imaginary time formalisms were investigated and the usual fluctuation dissipation results were recovered.	This is a new approach and had previously not been derived for the macroscopic QED model of [84]. Future work could include the completion of the application of the Schwinger-Keldysh method in order to attempt to complete the formalism for arbitrary temperatures and time dependencies.

Appendix A

Principal parts and the Small Imaginary Shift $\mathrm{i}\eta$

A.1 Imaginary shift $\mathrm{i}\eta$

In the main text, it was stated that the small imaginary number $\mathrm{i}\eta$ picks which half plane contains the poles of an integrand. For the Green's function of the harmonic oscillator G_ω solution to (III.19), this is equivalent to choosing either the retarded or the advanced Green's function as each is only analytic in the only either the upper half or lower half complex plane.

To understand this, consider the exponential term of (III.23) analytically continued into the complex plane. If one picks $t - t'$ to be strictly positive, when the imaginary part of the frequency Im$[\omega] > 0$, the exponential diverges as $|\omega| \to \infty$ whereas it converges to zero when Im$[\omega] < 0$. Conversely, if one picks $t - t' < 0$, then the opposite is true; the exponent converges to zero for large $|\omega|$ in the upper half plane and diverges in the lower half plane.

However, Cauchy's theorem states that the integral of an analytic function around a closed loop is equal to $2\pi\mathrm{i}$ times the sum of the residues within the loop. Thus when one evaluates integrals of the form (III.23), ones uses the convergence of the exponent to zero at infinity to write that the line integral is equal to the poles contained within the region bounded by the real axis and the convergent arc. For example, the arc converges in the upper half plane in Figure A.1 a).

In this case, if one takes the infinitesimal number $i\eta$ to be in the upper half plane, since the integral converges in the upper plane, it will pick out the poles and be non-zero. If the integral converges in the lower half plane (when $t - t' > 0$), the integral will not pick out the poles in the upper half plane and will be zero. If the η is taken to be negative, the opposite occurs. In this way, deciding which half plane the imaginary resides in sets whether we wish to keep the retarded solutions (non zero for $t - t' > 0$) or the advanced solutions (non zero for $t - t' < 0$).

A.2 Poles and Principal Parts

A useful identity that results from such considerations is the expansion of the frequency space oscillator Green's function via

$$\frac{1}{\omega_1^2 - (\omega \pm i\eta)^2} = \mathsf{P}\frac{1}{\omega_1^2 - \omega^2} \mp \frac{\mathrm{i}\pi}{2\omega}\left[\delta(\omega_1 - \omega) + \delta(\omega_1 + \omega)\right] \tag{A.1}$$

where P indicates the *principal part* of the integral containing the subsequent fraction. To understand the principal part, and eventually the identity (A.1), consider first the slightly simpler identity

$$\frac{1}{\omega_1 - \omega \pm \mathrm{i}\eta} = \mathsf{P}\frac{1}{\omega_1 - \omega} \mp \mathrm{i}\pi\delta(\omega_1 - \omega). \tag{A.2}$$

In both cases, the principal part is understood by considering the case where $\eta = 0$ from the outset [48] as in Figure A.1 b) for example. Here, the LHS of (A.2) will be multiplied by some function $f(\omega)$ which is analytic in the upper half plane. The pole in the fraction on the LHS now resides on the real axis and integrating along this is not defined. However, the contour shown skirts around the pole at ω by following an arc of radius δ into the plane in which the function $f(\omega)$ is analytic. Since the contour traced out contains no poles, one can write that $\oint d\omega f(\omega)/(\omega_1 - \omega) = 0$. If we consider $f(\omega) \to 0$ as $|\omega| \to \infty$ in the upper half plane, then one immediately writes

$$\int_{-\infty}^{\omega_1-\delta} d\omega\,\frac{f(\omega)}{\omega_1-\omega} + \int_{\omega+\delta}^{\infty} d\omega\,\frac{f(\omega)}{\omega_1-\omega} + \int_{C_\delta} d\omega\,\frac{f(\omega)}{\omega_1-\omega} = 0 \tag{A.3}$$

where C_δ is the arc contour of radius δ. The sum of the two first integrals is what is referred to as the *principal part* of the integral as introduced above. By evaluating the third term, one can establish the value of this principal integral and can be seen to be $\int_{C_\delta} d\omega\, f(\omega)/(\omega_1-\omega) = -\mathrm{i}\pi f(\omega_1)$. The sign of this part is determined by which plane the function $f(\omega)$ is analytic in. If it were analytic in the lower half plane instead, the contour would be that of Figure A.2 b) and the sign of the integral over the arc would reversed since the contour C_δ now runs in the anti-clockwise direction. In this sense, since the retarded Green's function in frequency space is analytic in the upper half plane and the advanced is analytic in the lower half plane, one can now see how the expression (A.2) works. For example, supposing one picks the small imaginary part to be in the upper half plane, this expression becomes

$$\frac{1}{\omega_1-\omega+\mathrm{i}\eta} = \mathsf{P}\frac{1}{\omega_1-\omega} - \mathrm{i}\pi\delta(\omega_1-\omega). \tag{A.4}$$

If one convolutes this with a function analytic in the upper half plane $f(\omega)$ (typically the *retarded* solutions), then the principal value part becomes $\mathrm{i}\pi f(\omega_1)$ and the two parts cancel leaving zero. However, if one convolutes it with the advanced solution $f^\star(\omega)$, then the principal changes sign and instead the result is $-2\pi\mathrm{i}f(\omega_1)$. The opposite is also true, and it is in this sense that this small imaginary part 'picks out' the advanced or retarded solutions. Expression (A.1) is just the case where there are two solutions, one positive and one negative, but the principal remains the same and an analogous explanation can be followed.

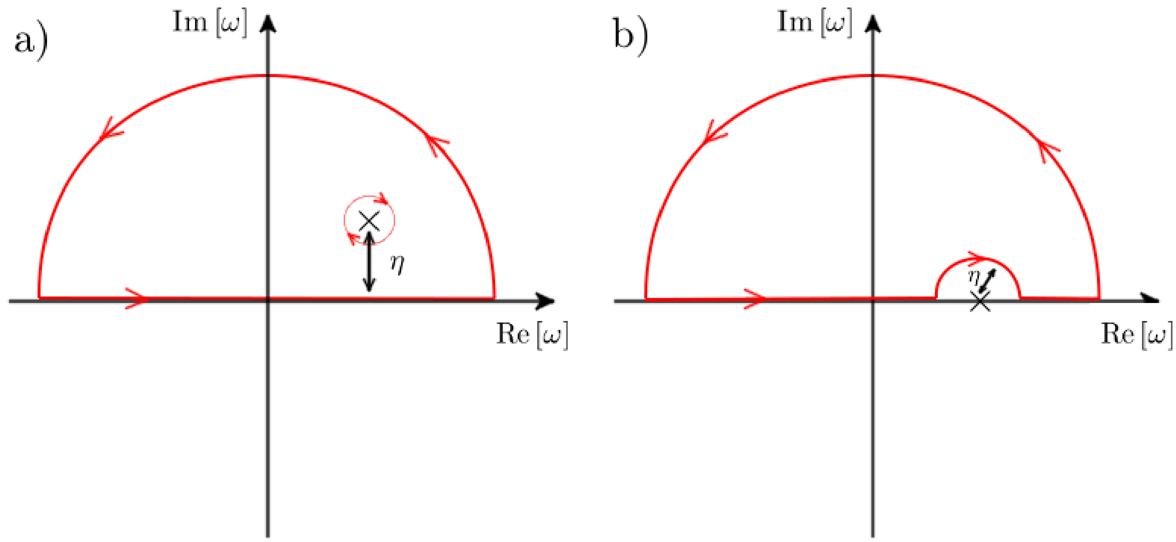

Figure A.1: a) Contour integral in upper half plane (convolution with $f(\omega)$), where the small imaginary part $i\eta$ is positive. The limit $\eta \to 0$ is understood to be taken right at the end of any calculation. b) The contour that avoids a pole on the real axis illustrating and enabling the calculation of the principal part of the integral that appears in the identity (A.1).

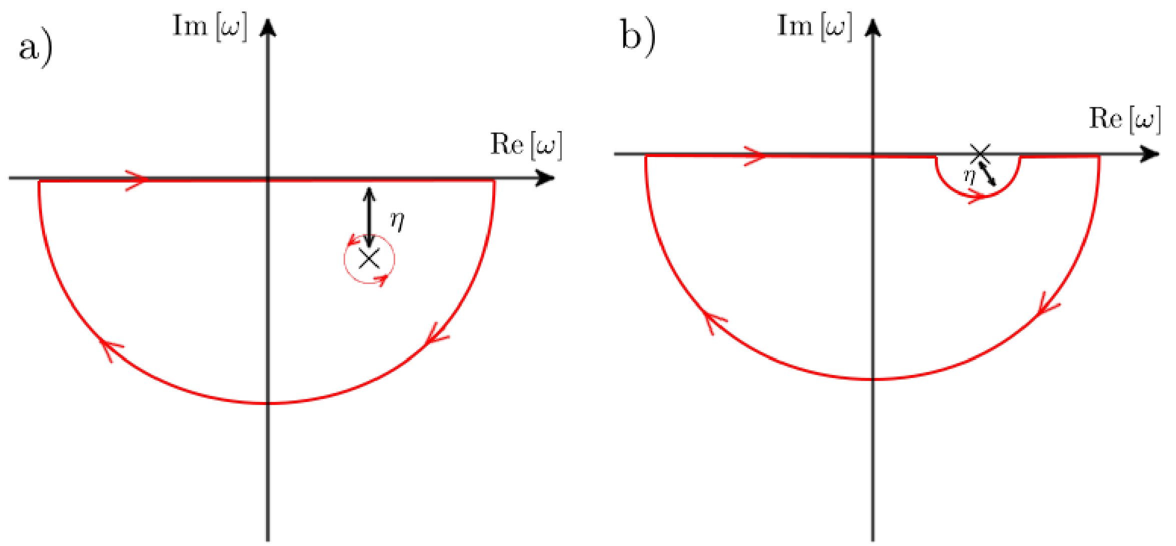

Figure A.2: a) Contour integral in lower half plane (convolution with $f^{\star}(\omega)$, the small imaginary part $-i\eta$ is negative. The limit $\eta \to 0$ is understood to be taken right at the end of any calculation. b) The contour that avoids a pole on the real axis illustrating and enabling the calculation of the principal part of the integral that appears in the identity (A.1) when one sets $\pm \to -$

.

Appendix B

Proof for Delta Function Identity

We wish to evaluate the limit

$$\lim_{T\to\infty}\frac{1}{T}\left|\int\limits_{-T/2}^{T/2} dt\ v\left(t\right) e^{i(\omega_1+\omega_2)t}\right|^2 \tag{B.1}$$

where $\mathbf{v}(t) = z_0\nu\cos(\nu t)\hat{\mathbf{z}}$. Performing the integral and taking the limit yields

$$\lim_{T\to\infty}\frac{1}{T}\left|\int\limits_{-T/2}^{T/2} dt\ v\left(t\right) e^{i(\omega_1+\omega_2)t}\right|^2 =$$
$$\frac{z_0^2\nu^2}{4}\left(2\pi\delta\left(\omega_1+\omega_2-\nu\right)+2\pi\delta\left(\omega_1+\omega_2+\nu\right)+8\lim_{T\to\infty}\frac{\cos\left[\nu T\right]-\cos\left[\left(\omega_1+\omega_2\right)T\right]}{\left[\left(\omega_1+\omega_2\right)^2-\nu^2\right]T}\right) \tag{B.2}$$

where the first two terms are derived in the same way as (III.132). The third can be evaluated by considering that it will eventually be convoluted with a function of ω_1 such as

$$\int\limits_{-\infty}^{\infty} d\omega_1\ f(\omega_1)\times\lim_{T\to\infty}\frac{\cos\left[\nu T\right]-\cos\left[\left(\omega_1+\omega_2\right)T\right]}{\left[\left(\omega_1+\omega_2\right)^2-\nu^2\right]T}. \tag{B.3}$$

Over the the range $(-\infty,\infty)$, only a small region will contribute to the integral when T is large, namely the region around $\nu = \pm(\omega_1+\omega_2)$. This integral is therefore equivalent to the following expression

$$\lim_{T\to\infty}\int\limits_{-\Delta}^{\Delta} d\eta\; f(\nu-\omega_2+\eta)\times\frac{\cos\left[\nu T\right]-\cos\left[\left(\nu+\eta\right)T\right]}{\left[\left(\nu+\eta\right)^2-\nu^2\right]T}. \tag{B.4}$$

For large T, Δ will be small and as such we can treat η as small and make the following approximation

$$\lim_{T\to\infty}\frac{1}{T}\left|\int\limits_{-T/2}^{T/2} dt\; v\left(t\right)e^{i(\omega_1+\omega_2)t}\right|^2 \approx \lim_{T\to\infty}\int\limits_{-\Delta}^{\Delta} d\eta\; f(\nu-\omega_2+\eta)\times\frac{\sin\left[\nu T\right]}{2\nu}. \tag{B.5}$$

In the limit $T\to\infty$, the range $\Delta\to 0$ and as such the area under the integral above vanishes. We can thus ignore the third term of equation (B.2).

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